INVOLUTIONS ON A BANACH ALGEBRA.

by

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PREFACE.

The study of Banach *-algebras is one of the most fruitful topics in the Theory of Banach algebras. With the exception of part of (0.2), the topics considered in this thesis are all inspired by results which have been obtained for Banach *-algebras.

The first part of the Introduction is mainly concerned with the statement of certain relevant results in the theory of "Numerical Range". As it was felt desirable to place the "square root lemma" due to Ford in a more general context, (0.2) is devoted to the study of solutions of equations involving analytic functions defined on Banach algebras.

In Chapter 1, a simple proof of a celebrated result (due to Kadison) characterising isometries between B^*-algebras is given, using simple techniques from the theory of "Numerical Range".

We study in Chapter 2 generalisations of many of the results in [20], 4.4, and also develop a theory of general representations on Hilbert space.

In Chapter 3, we study some properties of a Banach $\#$-algebra (i.e. a Banach algebra which possesses an anti-linear involution $x \to x^{\#}$) whose involutions satisfies one or more algebraic conditions.

Finally, Chapter 4 is essentially devoted to the study of Banach $\#$ - algebras whose involutions satisfy certain metrical conditions, the latter being suggested by some of the properties of B^* -algebras.

CONTENTS .

BASIC NOTATIONS

In this thesis, if reference is made to a result and the result is not prefaced by a section number, the result is contained within the section in which the reference is made.

We set out certain fundamental notations which are used in the sequel.

(1) $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers respectively. $\mathbb{R}^{++} = \{\lambda \in \mathbb{R} : \lambda > 0\}$ and $\mathbb{R}^{--} = \{\lambda \in \mathbb{R} : \lambda < 0\}$. If $B \subset \mathbb{R}^{++}$, we say that $B > 0$. N is the set of positive integers. If $A \subset \mathbb{C}$, $\mathrm{Re}A = \{\mathrm{Re}\,\lambda : \lambda \in A\}$ and $\bar{A} = \{\bar{\lambda} : \lambda \in A\}$. $\mathbb{C}^{\infty}$ denotes the extended complex plane.

(2) Let A and B be sets and K be a subset of A. Let T be a given mapping of A into B. Then $TK = \{Tk : k \in K\}$ and $T_{|K}$ is the restriction of T to K.

(3) Let A be a topological space and K be a subset of A. K^{-} denotes the closure of K in A. $C(A)$ is the algebra consisting of all continuous complex-valued functions defined on A, and $C_0(A)$ is that subalgebra of $C(A)$ consisting of all those elements in $C(A)$ which vanish at infinity. $C_b(A)$ is the subalgebra of $C(A)$ which consists of all the bounded functions in $C(A)$.

(4) Every linear space mentioned in the sequel will be assumed to be over the field $\mathbb{C}$ unless otherwise stated. If X is a linear space, A and B are non-empty subsets of X, and $\lambda \in \mathbb{C}$, we set $A + B = \{a + b : a \in A, b \in B\}$ and $\lambda A = \{\lambda a : a \in A\}$.

(5) Let X be a normed linear space with norm $\|\cdot\|$. X' is the

(continuous) dual of X, and $X'' = (X')'$, X' being, of course, regarded as a Banach space in its own right. $S(X)$ is the unit sphere and X_1 the unit ball of X. If $x \in S(X)$, $D_X(x) = \{f \in X' : f(x) = 1 = \|f\|\}$; when it is clear from the context that the space X is concerned, we shall normally write $D(x)$ in place of $D_X(x)$. If we wish to emphasise the dependence of $D(x)$ on the norm $\|\cdot\|$, we denote the latter set by $D_{\|\cdot\|}(x)$. $B(X)$ is the algebra of all bounded linear operators on X.

(6) Every algebra A considered in this thesis is assumed to be associative and over $\mathbb{C}$. Let A be an algebra and B a non-empty subset of A. If $x \in A$, $AxA = \left\{ \sum_{i=1}^{m} a_i x b_i : a_i, b_i \in A, m \in \mathbb{N} \right\}$ The meanings of Ax and xA are obvious.

If $x, y \in A$, we set $x \circ y = x + y - xy$. $Sp(x)$ denotes the spectrum of x in A. If B is a subalgebra of A and $x \in B$, $Sp_B(x)$ is the spectrum of x for the algebra B.

If $x \in A$ and $Sp(x)$ is a bounded, non-empty set, $|Sp(x)| = \sup\{|\lambda| : \lambda \in Sp(x)\}$. If x is an element of a subalgebra B of A, $|Sp_B(x)|$ will have an obvious meaning whenever it is defined. If A is a normed algebra, we write $\rho(a) = \lim_{n \to \infty} \|a^n\|^{1/n}$ $(a \in A)$. When A is a Banach algebra, it is well-known that $\rho(a) = |Sp(a)|$ for all $a \in A$.

Now, let F be a linear functional on an algebra A. In accordance with [5], we adopt the following notations :

$$N_F = \{x \in A : F(x) = 0\}, \qquad L_F = \{x \in A : Ax \subset N_F\}.$$
$$K_F = \{x \in A : xA \subset N_F\}, \qquad P_F = \{x \in A : AxA \subset N_F\}.$$

We put $X_F = A - L_F$, $Y_F = A - K_F$ and $Z_F = A - N_F$. $a \to T_a^F$

(resp. $a \to S_a^F$) is the left regular representation (resp. the right regular representation *) of A on X_F (resp. Y_F).

(7) In this thesis, a knowledge of the material presented in [20] will be assumed.

When we say that an algebra A is a unital Banach algebra, we mean that it is a Banach algebra with an identity element 1 such that $\|1\| = 1$.

Let A be a Banach algebra with identity element 1. Let $a \in A$; $C(a)$ is the closed subalgebra of A generated by a and 1, and $\{a\}_{cc}$ is the second commutant of $\{a\}$ in A.

If A is a commutative Banach algebra, Φ_A is the carrier space of A.

* Strictly speaking, of course, $a \to S_a^F$ is an anti-representation of A.

INTRODUCTION

(0.1) NUMERICAL RANGE AND OTHER TOPICS.

In recent years, a new approach has been developed to deal with certain situations in operator theory and the theory of Banach algebras. This approach investigates and exploits certain geometrical properties of elements in a Banach algebra and has been the means of proving some striking results in the theory of such algebras, in particular providing natural tools for studying B*-algebras.

The origins of this approach are to be found in the geneticalwork of Bohnenblust and Karlin ([1]) , in which is proved the important fact that, in a unital Banach algebra, the identity element is a vertex of the unit sphere.

The next significant advance was made by I. Vidav ([24]) who gave a metric characterisation of B*-algebras with identity (see (4.1));his results, however, fit naturally into the context of the theory of numerical range, the latter being introduced by G. Lumer in [15].

The classical notion of numerical range was, of course, formulated for operators on a Hilbert space, but Lumer showed that it is possible to introduce a fruitful generalisation of this notion within the much more general context of operators on a normed linear space, the inner product which functions in the classical case being replaced by a semi-inner-product on the normed linear space in question. Since 1961 , the theory of numerical range has been further developed by many mathematicians. (For details, see [6]).

In this section, we shall survey only those results in the above field which are relevant to this thesis; proofs of these results will normally be omitted. We commence by defining two basic geometrical concepts used in the above approach.

Let X be a Banach space. An element u of $S(X)$ is said to be a VERTEX of $S(X)$ if $\bigcap_{F \in D(u)} N_F = (0)$, i.e. $D(u)$ is a total family of linear functionals on X.

Let A be a unital Banach algebra. An element $x \in S(A)$ is said to be a UNITARY POINT if x^{-1} exists in A and $\|x\| = \|x^{-1}\| = 1$.

We now examine the notion of numerical range* which was introduced for an arbitrary normed space in [4]. This notion is closely related to the original concept of numerical range introduced by Lumer in [15].

Let X be a normed linear space and $T \in B(X)$. For $x \in S(X)$ let $V(T,x) = \{f(Tx) : f \in D(x)\}$ and $V(T) = \bigcup\{V(T,x) : x \in S(X)\}$. We say that $V(T)$ is the NUMERICAL RANGE of T. $V(T)$ has the following properties. (See [6], Chapter 3.)

THEOREM 1. $\sup \operatorname{Re} V(T) = \lim_{\alpha \to 0+} \alpha^{-1}\{\|I + \alpha T\| - 1\}$, where I is the identity element of $B(X)$. Also, if $v(T) = \sup\{|\lambda| : \lambda \in V(T)\}$ then $v(T) \leq \|T\| \leq e\,v(T)$.

If $T \in B(X)$ and $V(T) \subset \mathbb{R}$, we say that T is a HERMITIAN operator on X. We require the following theorem; the proof (which is easy) is omitted.

THEOREM 2. Let A be a subspace of $B(X)$ and suppose there exists a subset H of A consisting of hermitian operators in

*The motivation for this notion was an earlier paper by F. L. Bauer, who studied this concept for finite-dimensional normed spaces.

$B(X)$ such that $A = H + iH$. Then $A^- = H^- + iH^-$.

NOTE. In Theorem 2, H^- clearly consists of hermitian operators.

Now let A be a normed algebra and let $a \in A$. For $x \in S(A)$ let $V(a,x) = \{f(ax) : f \in D(x)\}$ and $V(a) = \bigcup\{V(a,x) : x \in S(A)\}$. Then $V(a)$ is said to be the NUMERICAL RANGE of a. (See [3]). For each $a \in A$, let $v(a) = \sup\{|\lambda| : \lambda \in V(a)\}$.

If $a \to T_a$ is the left regular representation of A, it is clear that $V(T_a) = V(a)$, thus relating the above two notions of numerical range. Now, for $a \in A$, let $\|a\|_1 = \sup\{\|ax\| : x \in S(A)\}$. Clearly, $\|\cdot\|_1$ is an algebra semi-norm on A which is majorised by $\|\cdot\|$.* Further, if $a \in A$ and $|\cdot|$ denotes the operator norm on $B(A)$, $|T_a| = \|a\|_1$. Using these observations, we have the following result:

THEOREM 3. Let A be a Banach algebra. Then $v(\cdot)$ is equivalent to the given norm $\|\cdot\|$ if and only if $\|\cdot\|_1$ is a complete norm on A.

We say that an element a of a normed algebra A is HERMITIAN if $V(a) \subset \mathbb{R}$. It is to be noted that in this thesis, the term "hermitian", when applied to an element of a normed algebra, will never be used to denote a self-adjoint element of a normed *-algebra unless the latter element is itself hermitian in the above sense.

Now let A be a unital Banach algebra. It is not difficult to show that $V(a) = V(a,1)$. (See [3]). The following results can be verified. (See [3], Theorem 3 and [1], Theorem 3.)

*i.e. $\|a\|_1 \leq \|a\|$ for all $a \in A$.

THEOREM 4. (1) Let $a \in A$. Then $Sp(a) \subset V(a)$ and

$$\sup \operatorname{Re} V(a) = \lim_{\alpha \to 0^+} \alpha^{-1}\{\|1 + \alpha a\| - 1\} .$$

(2) If u is a unitary point in A, u is a vertex of $S(A)$. In particular, 1 is a vertex of $S(A)$.

(3) An element $a \in A$ is hermitian if and only if

$$\lim_{\alpha \to 0} \alpha^{-1}\{\|1 + i\alpha a\| - 1\} = 0 \qquad (\alpha \in \mathbb{R}) .$$

COROLLARY. Let A be a B*-algebra and $a \in A$. Then a is a hermitian element if and only if $a^* = a$.

NOTE. The above Corollary is a consequence of [15], Theorem 21 .

We now state three results from the theory of Banach algebras. The first result is a deep result contained in [21] .

THEOREM 5. Let A be a B*-algebra with identity element 1 and ϕ be a linear mapping of A into a normed space B. Then $\|\phi\| = \sup\{\|\phi(u)\| : u \in U(A)\}$ *, where $U(A)$ is the set of unitary elements in A.

In [18], Palmer proves the following result which is a refinement of [21], Theorem 1 .

THEOREM 6. Let A be a B*-algebra with identity element and H be the set of self-adjoint elements in A. Let $A_e = \{e^{ih} : h \in H\}$. Then A_1 is the norm closure of the convex hull of A_e.

The following theorem, which was proved by Gleason ([10]) gives a characterisation of maximal ideals in commutative Banach algebras with identity.

*If ϕ is unbounded, this statement is to be interpreted as meaning that the set $\{\|\phi(u)\| : u \in U(A)\}$ is unbounded.

THEOREM 7. Let A be a commutative Banach algebra with identity element. A subspace X of A of codimension 1 in A is a maximal ideal if and only if X consists of singular elements.

We conclude this section by noting two simple results involving the Arens multiplications defined on the second dual of a Banach algebra.

(We recall that if A is a Banach algebra, the Arens multiplications "$\times$" and "$\circ$" on A'' are defined as follows. For $f \in A'$ and $y \in A$, define elements f_y and ${}_yf$ in A' by :

$$(f_y)(x) = f(xy) \quad , \quad ({}_yf)(x) = f(yx) \quad (x \in A).$$

For $\mu \in A''$ and $f \in A'$, define elements ${}_\mu f$ and f_μ in A' by :

$$({}_\mu f)(x) = \mu(f_x) \quad , \quad (f_\mu)(x) = \mu({}_xf) \quad (x \in A).$$

Then if $\mu, \nu \in A''$, the elements $\mu \times \nu$ and $\mu \circ \nu$ in A'' are defined by $\quad (\mu \times \nu)(f) = \nu({}_\mu f) \quad , \quad (\mu \circ \nu)(f) = \mu(f_\nu) \quad (f \in A').)$

LEMMA. Let A be a commutative Banach algebra. Then, for all $\mu \in A''$, $\mu \times \mu = \mu \circ \mu$.

Suppose now that A is a Banach $*$-algebra with continuous involution. For $f \in A'$, we define $f^* \in A'$ by $f^*(a) = \overline{f(a^*)}$ $(a \in A)$. Similarly for $\mu \in A''$, we define $\mu^* \in A''$ by $\mu^*(f) = \overline{\mu(f^*)} \quad (f \in A')$.

It is clear that the mapping $\mu \to \mu^*$ defines an anti-linear involution on A''. The following theorem relates the above involution on A'' to the multiplications "$\times$" and "$\circ$". (See [23], 2.10.)

THEOREM 8. Let $\mu, \nu \in A''$. Then $(\mu \times \nu)^* = \nu^* \circ \mu^*$.

(0.2) SOLUTIONS OF EQUATIONS INVOLVING ANALYTIC FUNCTIONS DEFINED ON BANACH ALGEBRAS.

Let g be an analytic function defined on a (non-empty) subset V of $\mathbb{C}$ and A be a Banach algebra with identity element 1. It is a well-known fact that, for an element x in A for which $Sp(x) \subset V$, we may specify in a natural way an element y in A which we can define to be $g(x)$.

In this section, we investigate some properties of those elements x in A which satisfy the equation $g(x) = a$, where a is an element in A. The considerations of this section are relevant, in particular, when A is a Banach *-algebra ; in fact, the "square-root lemma" of Ford ([8]) is a special case of a much more general result. (See the note after Theorem 5.)

The present study was motivated by the work of E. Hille which is contained in [12]. In the latter paper, Hille examines the equations $x^n = a$ and $e^y = b$ $(x,a,y,b \in A,\ n \in N)$, when these equations are meaningful.

We now give a synopsis of previous results which have been obtained by various mathematicians and which are relevant to the considerations of this section. Although Hille's results in [12] are the deepest and most general, we shall also mention related results which have been obtained by F.F.Bonsall (unpublished), L.T.Gardner ([9]) and J.W.M.Ford ([8]).

Throughout this section, A will denote a Banach algebra with identity element 1. We shall refer to the two equations considered by Hille :

$$x^n = a \quad \ldots(\alpha) \qquad , \qquad e^y = b \quad \ldots(\beta) \quad ,$$

where a and b are (sutiable) fixed regular elements in A, $x, y \in A$ and $n \in N$. As usual, a solution x of (α) (resp. (β)) will mean an element $x \in A$ (resp. $y \in A$) such that $x^n = a$ (resp. $e^y = b$). We have the following results :

(A) SPECIAL SOLUTIONS WITH COMMUTING PROPERTIES.

We ~~show~~ say that a solution x of (α) is a SPECIAL solution if $Sp(x)$ is irrotational (mod. $2\pi i/n$), i.e. $Sp(x) \cap Sp(\omega^i x) = \phi$ for $1 \leq i \leq n-1$, where $\omega = e^{2\pi i/n}$. A solution y of (β) is a SPECIAL solution if $Sp(y)$ is incongruent (mod. $2\pi i$), i.e. $Sp(y) \cap Sp(y + 2k\pi i) = \phi$ for $k = \pm 1, \pm 2, \ldots$.

Hille showed (see [12], Theorem 1 and Theorem 4) that if x (resp. y) is a special solution of (α) (resp. (β)), then x (resp. y) commutes with every other solution of (α) (resp. (β)).

(In a more special situation, F. F. Bonsall has obtained the following result : let $c \in A$ where $\rho(c) < 1$. Then there exists a unique element $x \in A$ such that $x \circ x = c$ and $\rho(x) < 1$. In this case, $x \in C(c)$.)

Hille also showed (see [12], Theorem 3) that there is a special solution for (α) if there exists a solution x of (α) for which there is a rectifiable arc C leading from 0 to a distant point of the plane such that $\omega^i C \cap Sp(x) = \phi$ for $1 \leq i \leq n$. An analogous result is true for the equation (β). (See [12], Theorem 6.)

(B) TOPOLOGICAL PROPERTIES OF ELEMENTS HAVING SPECIAL SOLUTIONS.

If x is a special solution of (α), Theorem 2 of [12] shows that there exists a neighbourhood V of x in A such that the set consisting of the $n^{th.}$ powers of elements in V is a neighbourhood of a . Similar results were also proved for the equation (β) . (See [12], Theorems 5 and 6 .)

(C) CHARACTERISATION OF SOLUTIONS IN TERMS OF SPECIAL SOLUTIONS.
Let x be a special solution and y be any solution of (α) . Then there exist idempotents $e_1, \dots, e_n$ in A such that, for $1 \leq i \leq n$, e_i commutes with both x and y , and $e_i e_j = \delta_{ij} e_i$ $(1 \leq i, j \leq n)$, $\sum_{i=1}^{n} e_i = 1$ and $y = x \sum_{i=1}^{n} \omega^i e_i$. (See [12], Theorem 1 .) An analogous result holds for the equation (β) . (See [12], Theorem 4 .)

(D) EXISTENCE AND UNIQUENESS OF SPECIAL SOLUTIONS.

(1) We have Bonsall's result. (See (A) .) Here, $1 - x$ is a special solution of the equation $z^2 = (1 - c)$ $(z \in A)$.

(2) In [9], Gardner proves the following result :
Let a be a regular element of A and suppose that $\mathrm{Sp}(a) \cap \mathbb{R}^- = \phi$. Then there exists a unique element $u \in A$ with the properties that $\mathrm{Re}\, \mathrm{Sp}(u) > 0$ and $u^2 = a$. (Here u is a special solution of the equation (α) with $n = 2$.)

(E) SOLUTIONS AND INVOLUTIONS.

We have Ford's square root lemma ([8]) :
Suppose that A is a Banach $*$-algebra and that a is a self-adjoint element in A such that $\rho(1 - a) < 1$. Then there exists a self-adjoint element x in A such that $x^2 = a$. (Again, x is a special solution of the equation (α) with $n = 2$.)

In this section, we shall examine a generalisation of the results contained in (A) , and shall show how the results contained in (D) and (E) fit easily into the general context evolved.

Generalisations of (B) and (C) will not be considered here.

In the sequel, the clause "let (g,V) be given" will mean: "let g be a function defined and analytic on a (non-void) open subset V of $\mathbb{C}$". We commence by proving two simple lemmas.

LEMMA 1. Let (g,V) be given and suppose that K is a compact subset of V such that g is one-to-one on K and $g'(k) \neq 0$ for each $k \in K$. Then there exists an open subset W of $\mathbb{C}$ such that $K \subset W \subset V$ and g is one-to-one on W.

PROOF: For each $n \in N$, let U_n be an open subset of $\mathbb{C}$ such that $K \subset U_n \subset V$ and $d(u,K) < 1/n$ for each $u \in U_n$. Suppose the result is false. Then there exist sequences $\{u_n\}$ and $\{v_n\}$ in $\mathbb{C}$, where, for each $n \in N$, $u_n, v_n \in U_n$, $u_n \neq v_n$ and $g(u_n) = g(v_n)$. It is clear that there exist $u, v \in K$ and a subsequence $\{n_k\}$ of N such that $u_{n_k} \to u$ and $v_{n_k} \to v$ as $n_k \to \infty$. It is also obvious that $g(u) = g(v)$. If $u \neq v$, the fact that g is one-to-one on K is contradicted. If, however, $u = v$, the fact that $g'(u) \neq 0$ is contradicted. The resulting contradiction proves the result.

LEMMA 2. Let U be a simply connected open subset of $\mathbb{C}$ and K be a compact subset of $\mathbb{C}$ such that $K \setminus U \neq \emptyset$. Then there exists a frontier point of K in $K \setminus U$.

PROOF: Suppose the result is false. Then $K \cup U$ is an open subset of $\mathbb{C}$, so that $L = \mathbb{C}^{\infty} \setminus (K \cup U)$ is a compact subset of $\mathbb{C}^{\infty}$. Now $K_1 = K \setminus U$ is compact in $\mathbb{C}^{\infty}$ and $L \cap K_1 = \emptyset$ and $\mathbb{C}^{\infty} \setminus U = L \cup K_1$. This contradicts the fact that U is simply connected.

Now, let (g,V) be given and suppose $x \in A$ such that $Sp(x) \subset V$. In the usual manner, we define

$$g(x) = (1/2\pi i)\int_{\Gamma}(\lambda 1 - x)^{-1} g(\lambda)\, d\lambda ,$$

where Γ is a ~~sutieble~~ oriented envelope for $Sp(x)$. (See [11], p. 104.) $g(x)$ is thus defined for those elements x in A for which $Sp(x) \subset V$. The following theorem is not difficult to verify. (c.f. [17], Theorem 7, subsection 6, §11.)

THEOREM 1. (1) $g(x) \in \{x\}_{CC}$, and if $\phi \in \Phi_{\{x\}_{CC}}$, $\widehat{g(x)}(\phi) = g(\hat{x}(\phi))$.

(2) If $g(\lambda) = \sum_0^\infty c_n \lambda^n$ for all $\lambda \in V$, where $\{c_n\} \subset \mathbb{C}$, then $g(x) = \sum_0^\infty c_n x^n$.

The following result is of importance in the sequel.

THEOREM 2. Let (g,V) and $(f,g(V))$ be given. Suppose $x \in A$ and $Sp(x) \subset V$. Then $f(g(x)) = (f(g))(x)$.

PROOF: The proof, which depends on an application of Fubini's theorem, is contained mutatis mutandis in the proof of [11], Theorem 5.17.8.

Now let (g,V) be given, a be a fixed element of A and $x \in A$ such that $g(x) = a$. Examination of (A) leads us to say that x is a SPECIAL SOLUTION* of the equation $g(z) = a$ $(z \in A)$ if $g(\lambda)$ is one-to-one on $Sp(x)$ and $g'(\lambda) \neq 0$ for each $\lambda \in Sp(x)$. We have the following result.

THEOREM 3. Let (g,V) be given, $a \in A$ and x be a special solution of the equation $g(z) = a$ $(z \in A)$. Then $\{x\}_{CC} = \{a\}_{CC}$.

PROOF: By Lemma 1, there exists an open subset W of $\mathbb{C}$ such that

*This notion has also been introduced in [7], where Theorem 3 is essentially obtained.

$Sp(x) \subset W \subset V$ and g is one-to-one on W. Let $X = g(W)$ and f be the inverse mapping of $g_{|W}$. Then f is an analytic function defined on X and, for $\lambda \in X$ and $\mu \in W$, $g(f(\lambda)) = \lambda$ and $f(g(\mu)) = \mu$. Now, by Theorem 1,(1), $a \in \{x\}_{CC}$ and $Sp_{\{x\}_{CC}}(a) \subset X$. Hence $y = f(a)$ is defined and clearly $y \in \{a\}_{CC}$. The result is proved by showing $x = y$.

Since $Sp(y) \subset W$, $g(y)$ is defined, and

$$g(y) = g(f(a)) = (g(f))(a) = a = g(x) \quad \text{using Theorem 2.}$$

Thus $y = f(g(y)) = f(g(x)) = x$.

COROLLARY. (c.f. (A)). If $y \in A$ and $g(y) = a$, then $xy = yx$.

Bonsall's result (see (A)) suggests the following question : when does x in Theorem 3 belong to $C(a)$? The following theorem gives a partial answer to this question.

THEOREM 4. Let (g,V) be given where $g(V)$ is simply connected and g is one-to-one on V. Then if $x \in A$ and $g(x) = a$, $x \in C(a)$.

PROOF: Since $g(x)$ exists, $Sp(x) \subset V$ and hence $Sp(a) \subset g(V)$. Using Lemma 2 and [20], Theorem 1.6.12, we have $Sp_{C(a)}(a) \subset g(V)$. The rest of the proof follows as in Theorem 3.

From the above considerations, we see that if (g,V) is given such that $g'(\lambda) \neq 0$ for all $\lambda \in V$ and $a \in A$ for which $Sp(a) \subset g(V)$, the number of special solutions of the equation $g(z) = a$ $(z \in A)$ is precisely the number of distinct compact subsets of $g^{-1}(Sp(a))$ which are mapped in a one-to-one manner onto $Sp(a)$. Using this observation, it is very easy to prove the results of Gardner and Bonsall. (See (D) and (A).)

We conclude this section by examining briefly the equation $g(z) = a$ $(z, a \in A)$ when A is a Banach *-algebra. In this context we require that $\bar{V} = V$ and that g be SELF-ADJOINT on V, i.e. for each $\lambda \in V$, $\overline{g(\bar{\lambda})} = g(\lambda)$. (See [20], p. 304.) We then have the following result :

THEOREM 5. Let (g, V) be given, $\bar{V} = V$ and g be self-adjoint on V. Let A be a Banach *-algebra with continuous involution. Then if $x \in A$ and $g(x) = a$, $g(x^*) = a^*$.

PROOF: This result is clear when we consider finite sums of elements in A converging to $(2\pi i)^{-1}\int_C (\lambda 1 - x)^{-1} g(\lambda)\, d\lambda$, C being a suitable oriented envelope for $Sp(x)$ in V.

COROLLARY. Suppose that, in Theorem 5, g is one-to-one on V and $a^* = a$. Then $x^* = x$.

NOTE. If, in Theorem 5 and its corollary, g is a polynomial, we see (using Theorem 1, (2)) that the requirement that the involution in A be continuous is unnecessary. It is easy to prove Ford's square root lemma using this observation and the above corollary.

1. ISOMETRIES BETWEEN B*-ALGEBRAS.

(1.1) ISOMETRIES BETWEEN B*-ALGEBRAS .

In this chapter, we examine a celebrated result of Kadison ([14]) which characterises isometries between two B*-algebras each of which contains an identity element. We shall show that, using elementary concepts from the theory of numerical range (see (0.1)) it is possible to give an extremely simple, intrinsic proof of Kadison's result. (The question of the possibility of such a "numerical range" proof of the above result was posed by G. Lumer at the North British Functional Analysis Seminar which was held at Edinburgh in April, 1968. Lumer showed that such a proof can be given when the algebras concerned are commutative.)

The contents of this chapter are contained in [19]. In the sequel, A and B are B*-algebras, each containing an identity element 1, and T is a linear isometry mapping A onto B. $H(A)$ and $H(B)$ denote the sets of self-adjoint elements of A and B.

Kadison's theorem is now stated :

THEOREM. There exists a C*-isomorphism τ mapping A onto B (i.e. τ is a linear isomorphism which preserves self-adjoints and power structure) and a unitary element $v \in B$ such that $T = v\tau$.

The following three lemmas will be used in proving the above theorem.

LEMMA 1. Let v be an extreme point of B_1. Then v^*v is an idempotent.

PROOF: The proof, which is not difficult, is contained in the proof of [14], Theorem 1.

LEMMA 2. Let u be a unitary element of A. Then Tu is not a divisor of zero in B.

PROOF: Suppose $x \in B$ and $(Tu)x = 0$. Since T maps A onto B, there exists $y \in A$ such that $x = (Ty)^*$. Hence

$(Tu)(Ty)^* = 0 = (Ty)(Tu)^*$. Let $\alpha \in \mathbb{C}$. Then

$$\|u + \alpha y\|^2 = \|Tu + \alpha Ty\|^2 = \|(Tu + \alpha Ty)((Tu)^* + \bar{\alpha}(Ty)^*)\|$$
$$= \|(Tu)(Tu)^* + |\alpha|^2(Ty)(Ty)^*\|$$
$$\leq \|Tu\|^2 + |\alpha|^2\|Ty\|^2 .$$

This gives $\|u + \alpha y\| \leq (1 + |\alpha|^2 k)^{\frac{1}{2}}$, where $k = \|Ty\|^2$. It follows that as $\alpha \to 0$ with $\alpha \in \mathbb{R}$, we have both

$$\|1 + \alpha u^*y\| \leq 1 + o(\alpha) \quad , \quad \|1 + i\alpha u^*y\| \leq 1 + o(\alpha) .$$

Therefore $u^*y \in H(A) \cap iH(A) = (0)$, using (0.1), Theorem 4, Corollary. Since u^* is a regular element of A, $y = 0$, and hence $x = (Ty)^* = 0$. Thus, Tu is not a left divisor of zero in B, and it may be similarly shown that Tu is not a right divisor of zero in B.

LEMMA 3. Let u be a unitary element of A. Then Tu is a unitary element of B.

PROOF: Since T is a linear isometry of A onto B, T maps the extreme points of A_1 onto the extreme points of B_1. By (0.1), Theorem 4, (2) u is a vertex of $S(A)$ and hence an extreme point of A_1. Thus Tu is an extreme point of B_1.

Let $p = (Tu)^*(Tu)$. It follows easily from Lemma 2 that p is not a divisor of zero in B. Since by Lemma 1, p is an idempotent in B, we have $p(p - 1) = 0$. Hence $p = 1$, and

$(Tu)^*(Tu) = 1$.

Now, if $y \in B$ is an extreme point of B_1 , y^* is also an extreme point of B_1 . Hence $(Tu)^*$ is an extreme point of B_1 . Applying the above arguement to $(Tu)^*$ instead of to Tu , it is clear that $(Tu)(Tu)^* = 1$. Hence Tu is a unitary point of B .

PROOF OF THE THEOREM: By Lemma 3 , $T(1)$ is a unitary element of B . Let $v = T(1)$, and define the mapping τ of A into B by $\tau = v^*T$. τ is clearly a linear isometry mapping A onto B , and $\tau(1) = 1$.

Since $T = v\tau$, the theorem will be proved if we can show that τ is a C*-isomorphism . Let $h \in H(A)$. Then

$$\| 1 + i\alpha\tau(h) \| \leq \| 1 + i\alpha h \| \leq 1 + o(\alpha) \quad (\alpha \in \mathbb{R}, \alpha \to 0.)$$

Thus, by (0.1), Theorem 4, Corollary, $\tau(h) \in H(B)$ and τ is a *-mapping.

It remains to show that $\tau(x^2) = (\tau(x))^2$ $(x \in A.)$ Let $h \in H(A)$ and $\alpha \in \mathbb{R}$. Then $e^{i\alpha h}$ is a unitary element of A . By Lemma 3 applied to τ, $\tau(e^{i\alpha h})$ is a unitary element of B , i.e. $\tau(e^{i\alpha h})\tau(e^{-i\alpha h}) = 1$. Thus

$$[1 + i\alpha\tau(h) - \alpha^2\tau(h^2)/2][1 - i\alpha\tau(h) - \alpha^2\tau(h^2)/2] = 1 + O(\alpha^3)$$

as $\alpha \to 0$, using the fact that τ is continuous and $\tau(1) = 1$.

Hence, $1 + \alpha^2[(\tau(h))^2 - \tau(h^2)] = 1 + O(\alpha^3)$ as $\alpha \to 0$; i.e. $[\tau(h)]^2 - \tau(h^2) = O(\alpha)$ as $\alpha \to 0$. Thus $[\tau(h)]^2 = \tau(h^2)$.

Now let $x \in A$ where $x = h + ik$ $(h, k \in H(A))$. Since $[\tau(h + k)]^2 = \tau[(h + k)^2]$, we have

$$\tau(hk + kh) = \tau(h)\tau(k) + \tau(k)\tau(h) \quad .$$

Hence $\tau(x^2) = \tau(h^2 - k^2 + i(hk + kh)) = [\tau(x)]^2$.

NOTE. The converse of the above theorem was also proved in [14] by Kadison, i.e. if τ is a C*-isomorphism mapping A onto B and v is a unitary element in B, then $T = v\tau$ is a linear isometry of A onto B. This result is an easy consequence of (0.1), Theorem 5.

To the author's knowledge, the study of isometries between general B*-algebras has not been prosecuted to date. The techniques employed above do not seem to be relevant to the above study; for example, the unit ball of a B*-algebra A possesses extreme points if and only if A has an identity element. (See [16].) It is easy (using the Banach-Stone theorem) to characterise isometries between commutative B*-algebras, but the author has been unable to characterise isometries in the general case.

2. REPRESENTATIONS ON NORMED SPACES.

(2.1) REPRESENTATIONS ON NORMED SPACES.

It is well-known that one of the most important methods used to investigate the properties of a Banach *-algebra A is that which examines the *-representations of A on Hilbert space. In [20], 4.4, Rickart sets out some of the general properties of such representations. We give a brief synopsis of the results described by Rickart in the above section. Throughout this synopsis, A is a *-algebra.

After the introduction of certain basic concepts (such as the notion of an essential *-representation of A on Hilbert space), Lemma 4.4.1 is proved, in which it is shown that if $a \to T_a$ is an essential *-representation of A on a Hilbert space H, then $f \in \{T_a f : a \in A\}^-$ for each $f \in H$. After deducing an easy corollary from this result, Rickart describes (in 4.4.3, 4.4.4 and 4.4.5) some of the properties of a pair of unitarily equivalent *-representations of A on Hilbert space.

Lemma 4.4.6 proves that if A is a normed algebra and $a \to T_a$ is a continuous *-representation of A on Hilbert space, then $|T_h| \leq \rho(h)$ for every self-adjoint element h in A, a fact which is used to show that the direct sum of a family of *-representations of a Banach *-algebra on Hilbert space is always defined.

Theorem 4.4.8 proves that any essential *-representation of A on Hilbert space is unitarily equivalent to a direct sum of (topologically) cyclic representations.

Finally, after a short discussion of the *-radical of a *-algebra, it is shown in Theorem 4.4.12 that a *-subalgebra A of $B(H)$ (where H is a Hilbert space) is (topologically) irreducible on H if and only if the centraliser of A in $B(H)$ consists entirely of scalar multiples of the identity operator.

Examination of many of the facts described above indicates the possibility that results similar to these may be true in more general circumstances. In this section, we examine this possibility and, in fact, obtain generalisations of the following results: 4.4.1, 4.4.2, 4.4.3, 4.4.4, 4.4.5, 4.4.8, and 4.4.12. The remaining results set out in [20], 4.4 are, perhaps, too deeply dependent on involutory properties to admit the type of generalisation considered in this section. In this section, our terminology will, in general, conform to that used in [20], 4.4 ; the terms "cyclic" and "irreducible" will, in particular, mean "topologically cyclic" and "topologically irreducible" respectively.

Let $a \rightarrow T_a$ be a representation of an algebra A on a normed space X. We prove generalisations of 4.4.1 and 4.4.2 when X is a normed reflexive space, 4.4.3, 4.4.4, and 4.4.5 when X is a Banach space, and 4.4.8 and 4.4.12 when X is a Hilbert space.*

The representation $a \rightarrow T_a$ is said to be ESSENTIAL if the set $\{x \in X : T_a x = 0 \text{ for every } a \in A\} = \{0\}$. If $a \rightarrow T_a$, restricted to an invariant subspace M of X, is cyclic, then M is called a CYCLIC SUBSPACE of X with respect to the representation. If $x \in X$, $X_x = \{T_a x : a \in A\}^-$. (It is to be noted that all the representations considered in this thesis are assumed to be normed

* The author is indebted to Professor F. F. Bonsall for suggesting that some of the above results, all of which were originally obtained when X is a Hilbert space, might allow the further generalisations indicated above.

representations. (See [20], p. 4.))

If $a \to \tau_a$ is a representation of A on a normed space Y, $a \to T_a$ and $a \to \tau_a$ are said to be ISOMETRICALLY EQUIVALENT provided there exists a linear isometry U mapping Y onto X such that $\tau_a = U^{-1} T_a U$ for every $a \in A$.

We note that if, for $a \in A$, $(T_a)^*$ is the adjoint operator of T_a defined on X', the mapping $a \to (T_a)^*$ of A on X' is an anti-representation. (If X is a Hilbert space, $(T_a)^*$ preserves its usual meaning. In this case, we shall still call the mapping $a \to (T_a)^*$ an anti-representation, although it is an anti-linear mapping.) For $f \in X'$, we define $X'_f = \{(T_a)^* f : a \in A\}^-$.

Let $E \subset X$ and $F \subset X'$. We denote

$$E^{\perp} = \{f \in X' : E \subset N_f\} , \quad F_{\perp} = \bigcap \{N_f : f \in F\} .$$

We note that if E is a closed subspace of X, then $(E^{\perp})_{\perp} = E$.

Motivated by [20], 4.4.1, we now examine the following question: under what conditions does $x \in X_x$ for each $x \in X$? It is easy to show that the condition that $a \to T_a$ be essential on X is in general not sufficient for this result to be true. The concept introduced in the following definition is of importance in answering the above question:

DEFINITION 1. The representation (or anti-representation) $a \to T_a$ of A on the normed space X is said to be STRONGLY ESSENTIAL on X if, for each $x \in S(X)$ and each $f \in D_X(x)$, there exists $a \in A$ such that $f(T_a x) \neq 0$.

It is obvious that if $a \to T_a$ is strongly essential on X, then it is also essential on X. If X is a Hilbert space, the mapping $a \to T_a$ is strongly essential on X if

$\{(T_a x, x) : a \in A\} = (0)$ implies that $x = 0$. If, in addition, the set $\{T_a : a \in A\}$ is a self-adjoint algebra of operators on X, $a \to T_a$ is strongly essential on X if and only if it is essential on X.

The following lemma gives an alternative characterisation of the notion of strong essentiality. The proof is omitted.

LEMMA 1. Let $a \to T_a$ be a representation of A on the normed space X. Then the representation $a \to T_a$ is strongly essential on X if and only if, for each $x \in S(X)$, there exists $a \in A$ such that $\|x - T_a x\| < 1$.

In the following three results, $a \to T_a$ is a representation of the algebra A on a normed reflexive space X. We show firstly that, with respect to strong essentiality on their respective spaces, the mappings $a \to T_a$ and $a \to (T_a)^*$ are closely related. The proof is omitted.

LEMMA 2. The representation $a \to T_a$ is strongly essential on X if and only if the anti-representation $a \to (T_a)^*$ is strongly essential on X'.

We adopt the following notation : if $\{K_\lambda : \lambda \in \Lambda\}$ is a family of subspaces of a topological vector space Z, $S(K_\lambda : \lambda \in \Lambda)$ is the smallest closed subspace in Z containing every K_λ.

LEMMA 3. Let K be a closed subspace of X' which is invariant with respect to the anti-representation $a \to (T_a)^*$. Then, if the representation $a \to T_a$ is strongly essential on X,

$$S((T_a)^* K : a \in A) = K .$$

PROOF: Suppose that $S = S((T_a)^* K : a \in A) \neq K$. Using the

Hahn-Banach theorem and the fact that S is a closed proper subspace of K, there exists g in $S(K')$ such that g vanishes on S.

Since K is a normed reflexive space in its own right, there exists k_1 in $S(K)$ such that $g(k_1) = 1$. By the Hahn-Banach theorem, we can extend g to a linear functional g_1 on X' such that $g_1 \in S(X'')$. We then have $g_1(k_1) = 1 = \|g_1\|$. Since $X'' = X$, there exists $x \in S(X)$ such that $g_1 = \hat{x}$. It is clear that $k_1 \in D_X(x)$.

Now, for each $a \in A$,

$$(0) = g_1((T_a)^*K) = \hat{x}((T_a)^*K) = \{k(T_a x) : k \in K\} .$$

Hence $k_1(T_a x) = 0$ for every $a \in A$. Since $k_1 \in D_X(x)$, this contradicts the fact that the representation $a \to T_a$ is strongly essential on X.

It is easily verified that if M is an invariant subspace of X' for the representation $a \to T_a$, $M^\perp$ is an invariant subspace of X' for the anti-representation $a \to (T_a)^*$.*

THEOREM 1. The representation $a \to T_a$ is strongly essential on X if and only if, for each $x \in X$, $x \in X_x$.

PROOF: If $x \in X_x$ for each $x \in X$, it is easy to show that the representation $a \to T_a$ is strongly essential on X.

Suppose, then, that $a \to T_a$ is a strongly essential representation on X. Clearly, $X_x^\perp$ is an invariant subspace of X' with respect to the anti-representation $a \to (T_a)^*$ for each $x \in X$. Let $x \in X$. Then, by Lemma 3, $S((T_a)^* X_x^\perp : a \in A) = X_x^\perp$.

* In an obvious way, we extend terminology, originally formulated for representations, to cover the case of anti-representations.

Now, if $a \in A$ and $f \in X_x^{\perp}$, $((T_a)^*f)(x) = f(T_a x) = 0$. Hence $x \in [S((T_a)^* X_x^{\perp} : a \in A)]_{\perp} = (X_x^{\perp})_{\perp} = X_x$. Thus the result follows.

COROLLARY. If $a \rightarrow T_a$ is strongly essential on X, then every non-zero closed invariant subspace of X contains a non-zero cyclic subspace.

NOTE. If A is a $*$-algebra, X is a Hilbert space and $a \rightarrow T_a$ is a $*$-representation of A on X, it is easily verified that the above theorem reduces to [20], 4.4.1.

We now examine analogues of [20], 4.4.3, 4.4.4 and 4.4.5.

THEOREM 2. Let $a \rightarrow T_a$ and $a \rightarrow \tau_a$ be two cyclic representations of an algebra A on the Banach spaces X and Y respectively. Then these representations are isometrically equivalent if and only if there exist cyclic vectors $x_1 \in X$ and $y_1 \in Y$ such that $\|T_a x_1\| = \|\tau_a y_1\|$ for all a in A.

PROOF: We first suppose that the representations are isometrically equivalent under a linear isometry U of Y onto X. Let y_1 be any cyclic vector in Y and define $x_1 = Uy_1$. Then, for $a \in A$,
$$\|T_a x_1\| = \|T_a U y_1\| = \|U \tau_a y_1\| = \|\tau_a y_1\| .$$
This proves one part of the theorem, since x_1 is clearly a cyclic vector for the representation $a \rightarrow T_a$.

The proof of the other part of the theorem proceeds as in the latter part of the proof of [20], Theorem 4.4.3.

COROLLARY 1. If two representations of A, each of which is defined on a Banach space, are isometrically equivalent, and one of the representations is cyclic, then so is the other.

COROLLARY 2. Let $a \to T_a$ and $a \to \mathcal{T}_a$ be representations of A on the Banach spaces X and Y respectively for which there exist cyclic vectors x_1 and y_1 such that $\|T_a x_1\| = \|\mathcal{T}_a y_1\|$ for all $a \in A$; in addition, suppose that $\|y_1\| = 1$ and that the representation $a \to T_a$ is essential on X. Then $\|x_1\| = 1$, and there exists a linear isometry J mapping Y' onto X' with the following properties :

(1) For every $f \in Y'$, $(Jf)(T_a x_1) = f(\mathcal{T}_a y_1)$ for each $a \in A$.

(2) J maps $D_Y(y_1)$ onto $D_X(x_1)$.

PROOF: Using a device from the proof of Theorem 2, we can construct a linear isometry U from Y onto X, with the property that $U\mathcal{T}_a y_1 = T_a x_1$ for all $a \in A$. Let $x_2 = Uy_1$. Then $T_a x_2 = U\mathcal{T}_a y_1 = T_a x_1$ for $a \in A$. Then $T_a(x_1 - x_2) = 0$ for all $a \in A$. Since the representation $a \to T_a$ is essential on X, $x_1 = x_2$. As U is an isometry, $\|x_1\| = \|y_1\| = 1$.

Let $f \in Y'$. Define the linear functional Jf on X by $(Jf)(Uy) = f(y)$ for $y \in Y$. It is easily verified that J is a linear isometry mapping Y' onto X'. For $f \in Y'$ and $a \in A$, $(Jf)(T_a x_1) = (Jf)(U\mathcal{T}_a y_1) = f(\mathcal{T}_a y_1)$. Thus (1) is proved. It is easy to prove (2) using the basic definition of J.

NOTES. (1) If A is a *-algebra, X and Y are Hilbert spaces and $a \to T_a$ and $a \to \mathcal{T}_a$ are *-representations, Theorem 2 reduces to [20], Theorem 4.4.3. (The condition in Theorem 2 that $\|T_a x_1\| = \|\mathcal{T}_a y_1\|$ for every $a \in A$ is equivalent, in this case, to the corresponding condition given in [20], Theorem 4.4.3.)

(2) It is clear that a more general result than (2) of Corollary 2 holds, viz. if $y \in S(Y)$, then $JD_Y(y) = D_X(U(y))$.

(3) If, in Corollary 2, Y is a Hilbert space, consideration of the mapping U shows that X is also a Hilbert

space.* In this case, (2) of Corollary 2 reduces to $(T_a x_1, x_1) = (\tau_a y_1, y_1)$ for all $a \in A$. If, in addition, A is a *-algebra and $a \to T_a$ and $a \to \tau_a$ are *-representations, the above corollary reduces to [20], Corollary 4.4.5. (It is easy to verify that a cyclic *-representation on a Hilbert space is automatically essential.)

We now proceed to investigate a generalisation of [20], Theorem 4.4.8. For the remainder of this section, H will denote a Hilbert space and $a \to T_a$ will be a representation of A on H.

We introduce the following concept: let $\{K_\sigma : \sigma \in \Omega\}$ be a set of Hilbert spaces and let $a \to T_a$ be a representation of A on $H = \sum^{(2)} K_\sigma$. Suppose further, that for $\sigma \in \Omega$, P_σ is the natural projection of H on K_σ. Then the representation $a \to T_a$ is said to be a PROJECTIVE DIRECT SUM OF CYCLIC REPRESENTATIONS if it has the following properties:

(1) For each $\sigma \in \Omega$, the mapping $a \to P_\sigma T_{a|K_\sigma}$ is a representation of A on K_σ.

(2) For each $\sigma \in \Omega$, there exists a family of subspaces $\{H(\lambda;\sigma) : \lambda \in \Lambda(\sigma)\}$ of K_σ such that:

(a) $H(\lambda_1;\sigma)$ and $H(\lambda_2;\sigma)$ are orthogonal if $\lambda_1 \neq \lambda_2$.

(b) $S(H(\lambda;\sigma) : \lambda \in \Lambda(\sigma)) = K_\sigma$.

(c) For each $\lambda \in \Lambda(\sigma)$, there exists a strongly essential cyclic representation $a \to T_a^{\lambda(\sigma)}$ of A on $H(\lambda;\sigma)$, such that the direct sum of the representations $a \to T_a^{\lambda(\sigma)}$ $(\lambda \in \Lambda(\sigma))$ is defined, and is unitarily equivalent (in a natural way) to the rep-

* i.e. the norm on X is given by a Hilbert space topology on X.

resentation $a \to P_\sigma T_{a|K_\sigma}$ of A on K_σ.

NOTES. (1) The notion of a direct sum of general representations on Hilbert space is defined in an obvious way. In the above definition, K_σ is identified with a subspace of H in an obvious manner.

(2) Suppose that in the above definition, A is a *-algebra and $a \to T_a$ is a *-representation of A on H. Let $b \in A$ and $\sigma \in \Omega$. Using (1) of the above definition, $P_\sigma T_{b^*} P'_\sigma T_b K_\sigma = (0)$, where $P'_\sigma = I - P_\sigma$, I being the identity operator on H. Hence $(T_b K_\sigma, P'_\sigma T_b K_\sigma) = (0)$, so that $T_b K_\sigma \subset K_\sigma$. Thus $a \to T_{a|K_\sigma}$ is a *-representation of A on K_σ. Using this observation, it is easy to show, that, in a natural way, the representation $a \to T_a$ of A on H is unitarily equivalent to a direct sum of cyclic *-representations of A on Hilbert space. We now prove the following theorem :

THEOREM 3. Every strongly essential representation of an algebra A on Hilbert space is isometrically equivalent to a projective direct sum of cyclic representations.

PROOF: Let $a \to T_a$ be a strongly essential representation of A on a Hilbert space H.

We note that, if H^1 is a closed subspace of H which is invariant under the anti-representation $a \to (T_a)^*$, then the mapping $a \to PT_{a|H^1}$ is a strongly essential representation of A on H^1, where P is the natural projection of H on H^1.

Let Ω be the class of all ordinal numbers whose cardinal numbers are less than or equal to 2^β, where β is the cardinal number of the set of closed subspaces of H .

Using transfinite recursion, we define for each $\sigma \in \Omega$ a pair (H_σ, K_σ) , where H_σ and K_σ are closed subspaces of H satisfying the following conditions :

(1) H_σ is an invariant subspace of H with respect to the anti-representation $a \to (T_a)^*$. (Thus the mapping $a \to P_\sigma T_a|_{H_\sigma}$ is a strongly essential representation of A on H_σ , where P_σ is the natural projection of H on H_σ .)

(2) If $H_\sigma \neq (0)$, there exists a maximal family of non-zero, mutually orthogonal, cyclic subspaces $\{H(\lambda;\sigma) : \lambda \in \Lambda(\sigma)\}$ of H_σ for the representation $a \to P_\sigma T_a|_{H_\sigma}$ such that $K_\sigma = S(H(\lambda;\sigma) : \lambda \in \Lambda(\sigma))$. If $H_\sigma = (0)$, $K_\sigma = (0)$.

(3) If τ is an ordinal less than σ , $H_\sigma \subset K_\tau^\perp \cap H_\tau$.

(For convenience we shall denote $P_\sigma T_a|_{H_\sigma}$ by T_a^σ , where $a \in A$ and $\sigma \in \Omega$.) In connection with (2) , we note that if for $\sigma \in \Omega$, $H_\sigma \neq (0)$, there do exist maximal families of non-zero mutually orthogonal cyclic subspaces of H_σ with respect to the representation $a \to T_a^\sigma$; this follows using Theorem 1 . (See the proof of [20], Theorem 4.4.8 .) Hence, if $H_\sigma \neq (0)$ has been defined, it is always possible to obtain a subspace of H_σ which satisfies the conditions imposed on K_σ in (2) .

We now construct the pairs (H_σ, K_σ) $(\sigma \in \Omega)$ using transfinite recursion.

Let $H_0 = H$, and let K_0 be any subspace of H satisfying (2) . This defines the first pair (H_0, K_0) .

Now let $\sigma \in \Omega$ and suppose that (H_τ, K_τ) has been defined in accordance with the above specified conditions for every ordinal τ

where $\tau < \sigma$.

If σ has a predecessor σ', define $H_\sigma = K_{\sigma'}^{\perp} \cap H_{\sigma'}$. In this case, since $H_{\sigma'}$ is an invariant subspace of the anti-representation $a \to (T_a)^*$ (by (1)), and as $K_{\sigma'}$ is invariant under the representation $a \to T_a^{\sigma'}$ (by (2)), H_σ is a closed invariant subspace of the anti-representation $a \to (T_a)^*$. Hence, the mapping $a \to T_a^{\sigma}$ is a representation of A on H_σ. If $H_\sigma = (0)$, the choice of $K_\sigma = (0)$ certainly satisfies all the required conditions. If, however, $H_\sigma \neq (0)$, let K_σ be any subspace of H_σ satisfying (2). Then (H_σ, K_σ) satisfies (1) and (2). If $\tau < \sigma'$, $H_\sigma \subset H_{\sigma'} \subset K_\tau^{\perp} \cap H_\tau$ by hypothesis; for σ' itself, $H_\sigma \subset K_{\sigma'}^{\perp} \cap H_{\sigma'}$ by construction. Hence (H_σ, K_σ) satisfies (3).

If σ has no predecessor, define $H_\sigma = \bigcap_{\tau < \sigma} H_\tau$. H_σ is clearly a closed subspace of H. By condition (1), each H_τ $(\tau < \sigma)$ is an invariant subspace for the anti-representation $a \to (T_a)^*$. Hence, H_σ satisfies condition (1). If $\tau < \sigma$, $H_\sigma \subset H_{\tau+1} \subset K_\tau^{\perp} \cap H_\tau$, so that (3) is satisfied. Finally, choosing K_σ to be a closed subspace of H_σ satisfying (2), we see that (H_σ, K_σ) satisfies all the requisite conditions. Thus the pairs (H_σ, K_σ) are defined for all $\sigma \in \Omega$.

Now, let $\sigma, \tau \in \Omega$, where $\tau < \sigma$; then $K_\sigma \subset H_\sigma \subset K_\tau^{\perp}$, so that if $K_\sigma \neq (0)$, $K_\sigma \neq K_\tau$ for $\tau < \sigma$. It follows that if $K_\sigma \neq (0)$ for every $\sigma \in \Omega$, then $K_\sigma \neq K_\tau$ whenever $\tau \neq \sigma$; however, if this were possible, the choice of Ω would be contradicted. Hence there exists $\sigma \in \Omega$ such that $K_\sigma = (0)$. Let α be the smallest ordinal in Ω such that $K_\alpha = (0)$, and let $\Phi = \{\tau \in \Omega : \tau < \alpha\}$. We now show that $\bigcap \{K_\tau^{\perp} : \tau \in \Phi\} = (0)$.

Let $x \in \bigcap\{K_{\tau}^{\perp} : \tau \in \Phi\}$. We prove using transfinite induction that $x \in H_{\tau}$ for each $\tau \in \Phi$. Clearly $x \in H_0$. Let $\sigma \in \Phi$ and suppose that $x \in H_{\tau}$ for every $\tau < \sigma$. If σ has a predecessor σ', we have $x \in H_{\sigma'} \cap K_{\sigma'}^{\perp} = H_{\sigma}$. If, on the other hand, σ has no predecessor, $x \in \bigcap_{\tau<\sigma} H_{\tau} = H_{\sigma}$. Thus $x \in H_{\sigma}$ for every $\sigma \in \Phi$.

Now, if α has a predecessor α', $\alpha' \in \Phi$ and $(0) = H_{\alpha} = H_{\alpha'} \cap K_{\alpha'}^{\perp}$, i.e. $K_{\alpha'} = H_{\alpha'}$. Hence $x \in K_{\alpha'} \cap K_{\alpha'}^{\perp} = (0)$. On the other hand, if α has no predecessor, $x \in \bigcap_{\tau<\alpha} H_{\tau} = H_{\alpha} = (0)$. Thus $\bigcap_{\tau \in \Phi} K_{\tau}^{\perp} = (0)$. Hence $S(K_{\sigma} : \sigma \in \Phi) = H$.

For $\sigma \in \Phi$, let $K_{\sigma} = S(H(\lambda;\sigma) : \lambda \in \Lambda(\sigma))$, where $\{H(\lambda;\sigma) : \lambda \in \Lambda(\sigma)\}$ is a maximal family of non-zero, mutually orthogonal subspaces of H_{σ} which are cyclic with respect to the representation $a \to T_a^{\sigma}$. Let $K_{\sigma}^1 = \sum^{(2)} H(\lambda;\sigma)$. As in [20], Theorem 4.4.8, there exists a natural isometry U_{σ} mapping K_{σ}^1 onto K_{σ}, given by $U_{\sigma} f = \sum_{\lambda \in \Lambda(\sigma)} f(\lambda)$, where $f \in K_{\sigma}^1$. For $\lambda \in \Lambda(\sigma)$, denote by $a \to \overset{\lambda}{T}{}_a^{\sigma}$ the restriction of the representation $a \to T_a^{\sigma}$ to the subspace $H(\lambda;\sigma)$.

Since for $a \in A$, $\lambda \in \Lambda(\sigma)$, $\|\overset{\lambda}{T}{}_a^{\sigma}\| \leq \|T_a\|$, the direct sum of the representations $a \to \overset{\lambda}{T}{}_a^{\sigma}$ is defined, and is a representation $a \to \tau_a^{\sigma}$ on K_{σ}^1. As in [20], Theorem 4.4.8, we have $U_{\sigma} \tau_a^{\sigma} = T_a^{\sigma} U_{\sigma}$ $(a \in A)$.

Now let H^1 be the direct sum of the Hilbert spaces $\{K_{\sigma}^1 : \sigma \in \Phi\}$. Since $H = S(K_{\sigma} : \sigma \in \Phi)$, it is clear that there exists an isometric mapping U of H^1 onto H given by $Uf = \sum_{\sigma \in \Phi} U_{\sigma} f(\sigma)$ $(f \in H^1)$. It is obvious that $U|_{K_{\sigma}^1} = U_{\sigma}$, where K_{σ}^1 is identified in an obvious way with a subspace of H^1. For

$a \in A$, define the operator τ_a on H^1 by $\tau_a = U^{-1}T_aU$. It is easily verified that $a \to \tau_a$ is a (strongly essential) representation of A on H^1 and is isometrically equivalent to the representation $a \to T_a$.

Let Π_σ be the natural projection of H^1 on K^1_σ for $\sigma \in \Phi$. To show that $a \to \tau_a$ is a projective direct sum of cyclic representations of A on H^1, we prove that, for $a \in A$, $\Pi_\sigma \tau_a | K^1_\sigma = \tau^\sigma_a$. Let $f \in K^1_\sigma$. Then, for $a \in A$,

$$\Pi_\sigma \tau_a f = \Pi_\sigma U^{-1} T_a U f = \Pi_\sigma U^{-1} T_a U_\sigma f = \Pi_\sigma U^{-1} T^\sigma_a U_\sigma f$$
$$= \Pi_\sigma U^{-1} U_\sigma \tau^\sigma_a f = \Pi_\sigma \tau^\sigma_a f = \tau^\sigma_a f .$$

The theorem is thus established.

NOTE. If A is a *-algebra and $a \to T_a$ is a *-representation of A on H, Theorem 3 reduces to [20], Theorem 4.4.8.

Finally we examine a generalisation of [20], Theorem 4.4.12. We require the following definitions :

DEFINITION 2. Let C be a subalgebra of a *-algebra B, where B contains an identity element 1. Then C is said to be an ANTI-SYMMETRIC SUBALGEBRA of B provided $C \cap C^* \subset \mathbb{C}1$.

DEFINITION 3. Let H be a Hilbert space and A be a subalgebra of $B(H)$. Denote by A^* the set $\{T^* : T \in A\}$. Then the pair of algebras, (A, A^*) is said to be MUTUALLY IRREDUCIBLE on H if there is no (non-zero) closed subspace of H which is invariant with respect to both A and A^*.

We now prove the following generalisation of [20], Theorem 4.4.12.

THEOREM 4. Let H be a Hilbert space and A any subalgebra of

$B(H)$. Then (A,A^*) is mutually irreducible on H if and only if the centraliser D of A in $B(H)$ is an anti-symmetric subalgebra of $B(H)$.

PROOF: Suppose firstly that (A,A^*) is mutually irreducible on H and that $D \cap D^* \not\subset \mathbb{C} I$, where I is the identity element of $B(H)$. Then there exists $T \in D$ such that $T \notin \mathbb{C} I$ and $T^* = T$.

Now let C be the $*$-subalgebra of $B(H)$ generated by the elements of $A \cup A^*$. Since (A,A^*) is mutually irreducible on H, C is an irreducible algebra of operators on H. Hence, by [20], Theorem 4.4.12, the centraliser of C in $B(H)$ is $\mathbb{C} I$.

But, since $T^* = T$, T commutes with all the elements of C, so that $T \in \mathbb{C} I$. This contradicts the choice of T. Hence D is an anti-symmetric subalgebra of $B(H)$.

Now suppose that (A,A^*) is not mutually irreducible on H. Examination of the first part of the proof of [20], Theorem 4.4.12 shows that, in this case, there exists a self-adjoint projection operator P, which is neither 0 not I, contained in D, so that D cannot be an anti-symmetric subalgebra of $B(H)$.

This concludes the proof of the theorem.

It is obvious that if $A = A^*$, the above theorem reduces to [20], Theorem 4.4.12.

(2.2) REPRESENTATIONS ON HILBERT SPACES.

In the standard theory of Banach *-algebras, there is a close relationship between positive functionals and *-representations on Hilbert space.

In a more general context in which the notion of a *-representation on a Hilbert space is replaced by that of a dual representation on a pair of Banach spaces in normed duality, F. F. Bonsall and J. Duncan show in [5] that there is a fundamental connection between the bounded linear functionals on and the dual representations of a Banach algebra.

In this section, we develop a representation theory which mediates between the special theory of *-representations of a Banach *-algebra and the general theory of dual representations of a Banach algebra, viz. a theory of representations of a Banach algebra on Hilbert space.

Our approach to this study starts out from what is essentially a special case of [5], Theorem 3. However, as the theory develops, the effect of the special properties of a Hilbert space becomes apparent. The linear functional relevant in this situation is that which we have called "the generalised positive functional".

Let X, Y and Z be vector spaces and suppose that J is a bilinear mapping of $Y \times X$ into Z, i.e. for $x_1, x_2 \in X$, $y_1, y_2 \in Y$ and $\lambda, \mu \in \mathbb{C}$, we have

$$J(y_1 + y_2, x_1 + x_2) = J(y_1, x_1) + J(y_1, x_2) + J(y_2, x_1) + J(y_2, x_2) ,$$

and $$J(\lambda y_1, \mu x_1) = \lambda\mu J(y_1, x_1) .$$

For convenience, if $x \in X$ and $y \in Y$, we shall write $y \circ x = J(y,x)$.

Now suppose that, in addition, there exists an anti-linear bijection $x \to x^{\#}$ of X onto Y. We say that a linear functional F defined on Z is a GENERALISED POSITIVE FUNCTIONAL with respect to $(X,Y,Z,\circ,\#)$ if and only if, for each $x \in X$, $F(x^{\#} \circ x) \geq 0$. The following simple lemma gives certain important elementary properties of generalised positive functionals : these properties are obvious analogues of corresponding properties of positive functionals on a *-algebra.

LEMMA 1. Let F be a generalised positive functional with respect to $(X,Y,Z,\circ,\#)$. Then, for $x_1, x_2 \in X$, we have $|F(x_1^{\#} \circ x_2)|^2 \leq F(x_1^{\#} \circ x_1)F(x_2^{\#} \circ x_2)$ and $F(x_1^{\#} \circ x_2) = \overline{F(x_2^{\#} \circ x_1)}$.

PROOF: The proof is formally identical to that used in [20], pp. 212 - 213 in connection with positive functionals.

Now, let A be an algebra and let F be a linear functional defined on A. We can define in a natural way a bilinear mapping "$\circ$" of $Y_F \times X_F \to Z_F$ given by $y' \circ x' = (yx)'$, where $y \in y' \in Y_F$, $x \in x' \in X_F$ and $(yx)' = yx + Z_F$. It is easily verified that "$\circ$" is a well-defined mapping. (See [5], section 4.) Let F' be the linear functional defined on the (one-dimensional) space Z_F by $F'(x') = F(x)$ $(x \in x' \in Z_F)$. We introduce the following definition :

DEFINITION 1. A linear functional F on A is said to be a GENERALISED POSITIVE FUNCTIONAL on A if there exists an anti-linear bijection $x' \to (x')^{\#}$ of X_F onto Y_F such that F' is a generalised positive functional with respect to $(X_F, Y_F, Z_F, \circ, \#)$. (i.e. $\langle x', (x')^{\#} \rangle_F \geq 0$ for all $x' \in X_F$, in the notation of [5].)

NOTE. It is easy to see that if A is a $*$-algebra, every positive functional on A is a generalised positive functional on A. The converse is obviously untrue.

We now give an example of a generalised positive functional on an algebra A which illustrates Definition 1.

EXAMPLE 1. Let $A = \left\{ \begin{bmatrix} x & \cdot \\ y & z \end{bmatrix} : x, y, z \in \mathbb{C} \right\}$; A is an algebra with the usual multiplication. We define the linear functional F on A by $F\left(\begin{bmatrix} x & \cdot \\ y & z \end{bmatrix} \right) = x + y + z$ $(x, y, z \in \mathbb{C})$. It is easily verified that $L_F = \left\{ \begin{bmatrix} \cdot & \cdot \\ b & -b \end{bmatrix} : b \in \mathbb{C} \right\}$ and $K_F = \left\{ \begin{bmatrix} b & \cdot \\ -b & \cdot \end{bmatrix} : b \in \mathbb{C} \right\}$. Let $u = \begin{bmatrix} x & \cdot \\ y & z \end{bmatrix}$ for some $x, y, z \in \mathbb{C}$. Define $u_1 = \begin{bmatrix} \frac{1}{2}\bar{x} & \cdot \\ \frac{1}{2}\bar{x} & \bar{y} + \bar{z} \end{bmatrix}$. Let $u' = u + L_F$ and $u_1'' = u_1 + K_F$. Then the mapping $u' \to u_1''$ is well-defined, and is an anti-linear bijection of X_F onto Y_F. Putting $(u')^{\#} = u_1''$ $(u \in A)$, it is easily verified that F is a generalised positive functional on A with respect to the mapping $u' \to (u')^{\#}$ $(u' \in X_F)$.

We require the following definition. (See [20], p. 213.)

DEFINITION 2. Let X, Y and Z be vector spaces, "$\circ$" represent a bilinear mapping of $Y \times X$ into Z and $x \to x^{\#}$ be an anti-linear bijection of X onto Y. Suppose that F is a generalised positive functional on Z with respect to $(X, Y, Z, \circ, \#)$ and that $a \to L_a$ is a representation of an algebra A on the space X. Then F is said to be ADMISSABLE with respect to the representation $a \to L_a$ if, for each $a \in A$, there exists a constant

$M_a > 0$ such that $F[(L_a u)^{\#} \circ (L_a u)] \leq M_a F(u^{\#} \circ u)$ for all $u \in X$. In particular, if F is a generalised positive functional on A, we say that F is ADMISSABLE on A if F is admissable with respect to the representation $a \to T_a^F$ of A on X_F.

The following three theorems deal with essentially the same situation, but for convenience, we have separated them.

With a few modifications, Theorem 1 is a special case of [5], Theorem 3. In the latter theorem, Bonsall and Duncan obtain some of the properties of a dual representation $a \to T_a$ of a Banach algebra on $(X, Y, \langle\, ,\, \rangle)$ (where X and Y are Banach spaces in normed duality under a bilinear form $\langle\, ,\, \rangle$) for which there exist topologically cyclic vectors x_0 in X and y_0 in Y. In Theorem 1 of the sequel, X and Y are each replaced by a Hilbert space H, the inner product on H taking over the role of the bilinear form $\langle\, ,\, \rangle$. In these results, we shall assume that A, H and the representation in question are proper.

THEOREM 1. Let $a \to T_a$ be a representation of an algebra A on a Hilbert space H, and suppose that x_0 and y_0 are topologically cyclic vectors in H for the representation $a \to T_a$ and the anti-representation $a \to (T_a)^*$ respectively. Then there exists a linear functional F defined on A with the following properties :

(1) $L_F = \{a \in A : T_a x_0 = 0\}$, $K_F = \{a \in A : (T_a)^* y_0 = 0\}$.

(2) There exists a linear (resp. anti-linear) monomorphism U (resp. V) mapping X_F (resp. Y_F) onto a dense subspace of H.

(3) For $x \in x' \in X_F$, $y \in y' \in Y_F$, $F'(y' \circ x') = (Ux', Vy')$,

where we are using notations defined earlier in the section.

PROOF: The proof is almost identical to that used in [5], Theorem 3. We define the linear functional F on A by
$F(a) = (T_a x_0, y_0)$ $(a \in A)$. U and V are defined by

$Ux' = T_x x_0$ $(x \in x' \in X_F)$ and $Vy' = (T_y)^* y_0$ $(y \in y' \in Y_F)$.

The rest of the proof is routine verification.

THEOREM 2. Let $a \to T_a$ be the representation of A discussed in Theorem 1. Then, in the notation of Theorem 1, we also have :

(1) There exist pre-Hilbert space topologies $\mathfrak{J}(X_F)$ and $\mathfrak{J}(Y_F)$ on the spaces X_F and Y_F respectively with respect to which U and V are isometries.

(2) Let X_F^- (resp. Y_F^-) denote the completion of X_F (resp. Y_F) with respect to $\mathfrak{J}(X_F)$ (resp. $\mathfrak{J}(Y_F)$)*. Then there exist an anti-linear isometry $u \to u^{\#}$ of X_F^- onto Y_F^- and a bilinear mapping "$\circ$" of $Y_F^- \times X_F^-$ onto Z_F such that F' is a generalised positive functional on Z_F with respect to $(X_F^-, Y_F^-, Z_F, \circ, \#)$.

PROOF: (1) For $x', y' \in X_F$, define $(x', y')_1 = (Ux', Uy')$. Since H is a Hilbert space and U is a linear monomorphism, it is clear that $(,)_1$ is an inner product on X_F, so that X_F becomes a pre-Hilbert space under the topology $\mathfrak{J}(X_F)$ induced on X_F by $(,)_1$. It is clear that U is a linear isometry from X_F into H when the former space is equipped with the norm derived from $\mathfrak{J}(X_F)$.

A similar arguement applies mutatis mutandis to Y_F and V.

* i.e. with respect to the norm on X_F (resp. Y_F) associated with $\mathfrak{J}(X_F)$ (resp. $\mathfrak{J}(Y_F)$).

It will always be assumed in the sequel that X_F and Y_F are equipped with the topologies $\mathfrak{J}(X_F)$ and $\mathfrak{J}(Y_F)$ respectively.

(2) Let $\bar{X_F}$ (resp. $\bar{Y_F}$) be the completion of X_F (resp. Y_F) with respect to the topology $\mathfrak{J}(X_F)$ (resp. $\mathfrak{J}(Y_F)$) . We denote each of the norms on $\bar{X_F}$ and $\bar{Y_F}$ by $\|\cdot\|$.

Since UX_F is dense in H and U is a linear isometry, U can be extended uniquely to a linear isometry $\bar{U}$ of $\bar{X_F}$ onto H. Similarly, V can be extended uniquely to an anti-linear isometry of $\bar{Y_F}$ onto H .

Now, for each $u \in \bar{X_F}$, let $u^{\#} = (\bar{V})^{-1}\bar{U}u$. It is easily verified that the mapping $u \to u^{\#}$ defines an anti-linear isometry of $\bar{X_F}$ onto $\bar{Y_F}$.

We now show that the mapping $(y',x') \to F'(y' \circ x')$ $(x' \in X_F$, $y' \in Y_F)$ of $Y_F \times X_F$ onto $\mathbb{C}$ is continuous. Let $\{x_n\}$ and $\{y_n\}$ be sequences in A for which $\lim_{n\to\infty} (y_n', x_n') = 0$ in $Y_F \times X_F$, where , for each $n \in N$, $y_n' \in Y_F$ and $x_n' \in X_F$. Then $y_n' \to 0$ in Y_F and $x_n' \to 0$ in X_F as $n \to \infty$. Now, for $n \in N$,

$$|F'(y_n' \circ x_n')| = |F(y_n x_n)| = |(T_{y_n} T_{x_n} x_0, y_0)|$$
$$= |(T_{x_n} x_0, (T_{y_n})^* y_0)| \leq \|x_n'\| . \|y_n'\| \quad .$$

Hence $\lim_{n\to\infty} F'(y_n' \circ x_n') = 0$, so that the mapping $(y',x') \to F'(y' \circ x')$ $(y' \in Y_F$, $x' \in X_F)$ is continuous. Hence the bilinear form $(y',x') \to F'(y' \circ x')$ $(y' \in Y_F$, $x' \in X_F)$ may be extended to a continuous bilinear form J mapping $\bar{Y_F} \times \bar{X_F}$ onto $\mathbb{C}$. Now, for $v \in \bar{Y_F}$ and $u \in \bar{X_F}$, we define $v \circ u$ to be that element x' of Z_F for which $J(v,u) = F'(x')$. Then "$\circ$" defines a bilinear mapping of $\bar{Y_F} \times \bar{X_F}$ onto Z_F .

Finally, we must show that F' is a generalised positive functional on Z_F with respect to $(X_F^-, Y_F^-, Z_F, \circ, \#)$.

Let $u \in X_F^-$ and $\{x_n\}$ and $\{y_n\}$ be sequences in A such that $\lim_{n\to\infty} x_n' = u$ and $\lim_{n\to\infty} y_n' = u^{\#}$, where $x_n' \in X_F$ and $y_n' \in Y_F$ for each $n \in N$. Then $\bar{U}u = \lim_{n\to\infty} T_{x_n} x_0$, $\bar{V}u^{\#} = \lim_{n\to\infty} (T_{y_n})^* y_0$ and $\bar{U}u = \bar{V}u^{\#}$.

Since the mapping $(v,u) \to F'(v \circ u)$ of $Y_F^- \times X_F^-$ onto $\mathbb{C}$ is continuous ,

$$F'(u^{\#} \circ u) = \lim_{n\to\infty} F'(y_n' \circ x_n') = \lim_{n\to\infty} F(y_n x_n)$$
$$= \lim_{n\to\infty} (T_{x_n} x_0, (T_{y_n})^* y_0) = \|\bar{U}u\|^2 \geq 0 .$$

This concludes the proof of the theorem.

COROLLARY. If x_0 and y_0 are cyclic vectors in H for the mappings $a \to T_a$ and $a \to (T_a)^*$ respectively, then F is a generalised positive functional on A .

If $a \to T_a$ is a representation of A on the Hilbert space H , we define $S = \{a \in A :$ there exists $b \in A$ such that $(T_a)^* = T_b\}$. If the above representation is, in addition, faithful, then S is a $*$-algebra in an obvious way.

THEOREM 3 . Let $a \to T_a$ be the representation of A in Theorem 2 . Then in the notations of Theorem 2 and the above, we have :

(1) For each $a \in A$, T_a^F (resp. S_a^F) is continuous on X_F (resp. Y_F) and so may be extended to a continuous linear operator $\hat{T}_a^F$ (resp. $\hat{S}_a^F$) on X_F^- (resp. Y_F^-).

(2) The representation $a \to \hat{T}_a^F$ on X_F^- is isometrically equivalent to the representation $a \to T_a$ on H .

(3) F' is admissable with respect to the representation $a \to \hat{T}_a^F$ on X_F^- .

(4) If $u \in X_F^-$ and $a, b \in S$ where $(T_b)^* = T_a$, then

$$(\hat{T}^F_a u)^{\#} = \hat{S}^F_b(u^{\#}) \quad .$$

(5) Let $a, b \in S$ where $(T_b)^* = T_a$. Then, if $x_0 = y_0$, $(a')^{\#} = b'$, where $a \in a' \in X_F$ and $b \in b' \in Y_F$.

PROOF: (1) Let $a \in A$ and suppose that $\{x_n\}$ is a sequence in A for which $x'_n \to 0$ as $n \to \infty$, where $x_n \in x'_n \in X_F$ for each $n \in N$. We show that $T^F_a x'_n \to 0$ as $n \to \infty$. By the continuity of $\bar{U}$, $T_{x_n} x_0 = \bar{U}x'_n \to 0$ as $n \to \infty$. Hence

$$\lim_{n\to\infty} \bar{U}(T^F_a x'_n) = \lim_{n\to\infty} T_a T_{x_n} x_0 = 0 \quad .$$

Thus $\lim_{n\to\infty} T^F_a x'_n = 0$, so that T^F_a is continuous on X_F . A similar arguement shows that S^F_a is continuous on Y_F .

(2) It is immediate that the mapping $a \to \hat{T}^F_a$ is a representation of A on X^-_F . We shall prove (2) by showing that, for each $a \in A$, $\bar{U}\hat{T}^F_a = T_a\bar{U}$.

Let $x \in x' \in X_F$. Then, for $a \in A$,

$$(\bar{U}\hat{T}^F_a)(x') = \bar{U}((ax)') = T_a(T_x x_0) = (T_a\bar{U})(x') \quad .$$

Using the facts that X_F is dense in X^-_F and that all of the mappings concerned are continuous, the required result follows.

(3) Fix $a \in A$ and let $u \in X^-_F$. Then , as in the proof of Theorem 2, (3) ,and using (2) of the present theorem,

$$F'\{(\hat{T}^F_a u)^{\#} \circ (T^F_a u)\} = \|\bar{U}(\hat{T}^F_a u)\|^2 = \|T_a(\bar{U}u)\|^2$$
$$\leq \|T_a\|^2 . \|\bar{U}u\|^2 = \|T_a\|^2 F(u^{\#} \circ u) \quad .$$

The required result now follows immediately.

(4) By an arguement similar to that used in (2) , it may be shown that $\bar{V}\hat{S}^F_a = (T_a)^*\bar{V}$ $(a \in A)$. Let $a, b \in S$ where $(T_b)^* = T_a$ and $u \in X^-_F$. Then

$$\bar{U}(\hat{T}^F_a u) = T_a(\bar{U}u) = T_a(\bar{V}u^{\#}) = [(T_b)^*\bar{V}](u^{\#}) = \bar{V}(\hat{S}^F_b u^{\#}) \quad .$$

Hence $(\hat{T}_a^F u)^{\#} = \hat{S}_b^F(u^{\#})$.

(5) Finally suppose $x_0 = y_0$ and $a,b \in S$ where $(T_b)^* = T_a$. Then, with $a' \in X_F$ and $b' \in Y_F$, $\bar{U}a' = T_a x_0 = (T_b)^* x_0 = \bar{V}b'$. Hence $(a')^{\#} = b'$ as required.

It is clear that, in the above theorem , if x_0 and y_0 are cyclic vectors in H for the mappings $a \to T_a$ and $a \to (T_a)^*$, F is an admissable generalised positive functional on A .

We now ask the following question : given a generalised positive functional F on an algebra A , when is F associated in a natural way with a representation of A on some Hilbert space?

The following theorem answers the above question . (See [20], Theorem 4.5.4 .)

THEOREM 4. Let A be an algebra and F be an admissable generalised positive functional on A . Then there exists a representation $a \to T_a$ of A on a Hilbert space H with the following properties :

(1) The kernel of the representation is P_F .

(2) There exists a linear mapping $x \to f_x$ of A onto a dense subspace of H such that

$$F'((x')^{\#} \circ (ax)') = (T_a f_x, f_x) \qquad (a,x \in A)^{*}. \cancel{x', (ax)' \in X_F).}$$

(3) If A has an identity element 1 , there exist vectors f and g in H which are topologically cyclic for the mappings $a \to T_a$ and $a \to (T_a)^*$ such that

$$F(a) = (T_a f, g) \qquad (a \in A) .$$

PROOF: For $x,y \in A$, $x',y' \in X_F$, define

$$(x',y') = F'((y')^{\#} \circ (x')) ,$$

* Here, primes denote the taking of canonical images in X_F .

where $x' \to (x')^{\#}$ is that anti-linear bijection mapping X_F onto Y_F which is associated with F. It is clear that $(,)$ defines an inner product on X_F; we denote by $\|\cdot\|$ the norm induced on X_F by $(,)$.

Since F is admissable on A, it follows that, for each $a \in A$, T_a^F is a bounded linear operator on X_F. Let H be the completion of X_F with respect to $\|\cdot\|$; then H is clearly a Hilbert space, and, for each $a \in A$, T_a^F can be uniquely extended to a continuous linear operator T_a on H. The mapping $a \to T_a$ is the required representation.

(1) If $a \in A$ and $T_a = 0$, then $T_a^F = 0$, i.e. $(T_a^F x', y') = 0$ for all $x', y' \in X_F$. Using this fact and the fact that the mapping $x' \to (x')^{\#}$ of X_F into Y_F is a surjection, we see that $F(yax) = 0$ for all $x, y \in A$. Hence $a \in P_F$. Since P_F is clearly contained in the kernel of the repesentation $a \to T_a$, the required result follows.

(2) For $x \in A$ let $f_x = x'$, where $x' \in X_F$. It is very easy to verify that the mapping $x \to f_x$ satisfies all the required properties.

(3) Finally, let A possess an identity element 1. In the notation of (2), it is clear that $f = f_1$ is a topologically cyclic vector in H for the representation $a \to T_a$.

Let $a \in A$ and $x', y' \in X_F$. Then

$$(T_a x', y') = F'((y')^{\#} \circ (ax)') = F(y_2 ax) \qquad (y_2 \in (y')^{\#})$$

$$= F'(S_a^F[(y')^{\#}] \circ x') = (x', (T_a)^* y') \quad .$$

Hence
$$[(T_a)^*(y')]^{\#} = S_a^F[(y')^{\#}] \quad .$$

Now define the vector g in X_F by $(g)^{\#} = 1'$ $(1 \in 1' \in Y_F)$.

Then $\{[(T_a)^*g]^{\#} : a \in A\} = \{S_a^F(1') : a \in A ,\ 1 \in 1' \in Y_F\} = Y_F$.
Hence $\{[(T_a)^*g] : a \in A\} = X_F$, so that g is a topologically cyclic vector in H for the anti-representation $a \rightarrow (T_a)^*$.

Lastly, if $a \in A$, $(T_a f, g) = F'[g^{\#} \circ (T_a f)] = F(a)$.

Let A be an algebra for which there exists an anti-linear injection $x \rightarrow x^{\#}$ of A into A ; let R be the range of this injection. A linear functional F on A is said to be a POSITIVE FUNCTIONAL on A if F is a generalised positive functional with respect to $(A, R, A, \times, \#)$, where "$\times$" represents the multiplication of A . (In more familiar language, this states that $F(x^{\#}x) \geq 0$ for all $x \in A$, the "$\times$" being omitted in accordance with the usual notation.)

Let F be a positive functional on A . Then for $x, y \in A$, we have by Lemma 1, $F(x^{\#}y) = \overline{F(y^{\#}x)}$. It follows that $(L_F)^{\#} \subset K_F$, where we are using an obvious notation.

Now suppose that the following condition holds :

$$R + K_F = A \qquad \ldots\ldots(1) \quad .$$

Let $x \in A$ and $x^{\#} \in K_F$. Then if $z \in A$, there exist (by (1)) $a \in A$ and $k \in K_F$ such that $z = a^{\#} + k$. Then $F(zx) = F(a^{\#}x) = \overline{F(x^{\#}a)} = 0$. Hence $x \in L_F$. Thus $(L_F)^{\#} = K_F \cap R$.
It follows that the mapping $x' \rightarrow (x^{\#})'$ $(x \in A ,\ x' \in X_F ;\ (x^{\#})' \in Y_F)$ is an anti-linear bijection of X_F onto Y_F .

Suppose now that G is a generalised positive functional on A , and that the mapping $x' \rightarrow (x')^{\#(G)}$ $(x' \in X_G)$ is the anti-linear bijection from X_G onto Y_G associated with G . G is said to be COMPATIBLE with respect to the (above) injection $x \rightarrow x^{\#}$ on A if, for each $x \in A$, $x^{\#} + K_G = (x + L_G)^{\#(G)}$.

The above discussion suggests the following question : if P is a set of generalised positive functionals on A, under what conditions does there exist a unique anti-linear injection mapping A into A with respect to which each element of P is compatible? This question is answered in the following theorem.

THEOREM 5. Let P be a (non-empty) set of generalised positive functionals defined on an algebra A; for each $F \in P$, denote by $x' \to (x')^{\#(F)}$ $(x' \in X_F)$ the anti-linear mapping of X_F onto Y_F associated with F. For $x \in A$, let

$$S(x) = \bigcap \{ (x + L_F)^{\#(F)} : F \in P \} .$$

Then there exists a unique anti-linear injection $x \to x^{\#}$ of A into A with respect to which each $F \in P$ is compatible if and only if :

(1) $S(x) \neq \phi$ for each $x \in A$.

(2) For $x \in A$, $S(x) = \{0\}$ if and only if $x = 0$.

PROOF: Suppose that there exists an anti-linear injection $x \to x^{\#}$ on A with respect to which each $F \in P$ is compatible. Let $x \in A$ and $F \in P$. Then $x^{\#} \in (x + L_F)^{\#(F)}$. Thus $x^{\#} \in S(x)$ and so $S(x) \neq \phi$. Further, if $S(x) = \{0\}$, $x^{\#} = 0$, so that $x = 0$. We show that when the injection $x \to x^{\#}$ is unique in the sense indicated in the statement of the theorem, $S(0) = (0)$.

Suppose that $S(0) \neq \{0\}$. Then there exists $k \neq 0$ in A such that $k \in \bigcap \{K_F : F \in P\}$. Let $M = \{x \in A : x^{\#} \in \mathbb{C} k\}$. Then M is a subspace of A of dimension at most one. Let ϕ be any non-zero anti-linear functional* on A which vanishes on M and, for $x \in A$, define $x^{*} = x^{\#} + \phi(x)k$. It is easily verified that

* i.e. for all $\lambda, \mu \in \mathbb{C}$, $x, y \in A$, $\phi(\lambda x + \mu y) = \bar{\lambda}\phi(x) + \bar{\mu}\phi(y)$.

the mapping $x \to x^*$ is anti-linear; further, if $x \in A$ and $x^* = 0$, then $x^{\#} = \phi(-x)k$. Thus $x^{\#} \in \mathbb{C}k$ and therefore $\phi(x) = 0$, so that $x^{\#}$, and hence x, is 0. Hence, the mapping $x \to x^*$ on A is an anti-linear injection.

Finally, for $x \in A$ and $F \in P$, $x^* + K_F = x^{\#} + K_F = (x')^{\#(F)}$, so that F is compatible with respect to the mapping $x \to x^*$. Since the mappings $x \to x^*$ and $x \to x^{\#}$ on A are not identical, the mapping $x \to x^{\#}$ cannot have the uniqueness property mentioned in the statement of the theorem. This proves the first part of the theorem.

Suppose now that P satisfies the conditions (1) and (2). Let $x, y \in A$. Then

$$\begin{aligned} S(x+y) &= \bigcap\left\{[x+y+L_F]^{\#(F)} : F \in P\right\} \\ &= \bigcap\left\{[x+L_F]^{\#(F)} + [y+L_F]^{\#(F)} : F \in P\right\} \\ &\supset S(x) + S(y), \end{aligned}$$

using the fact that both $S(x)$ and $S(y)$ are proper.

Since the mapping $x' \to (x')^{\#(F)}$ $(x' \in X_F)$ is anti-linear for each $F \in P$, it is clear that $S(\lambda x) = \bar{\lambda} S(x)$ $(\lambda \in \mathbb{C},\ x \in A)$.

Now, using (2), we have $(0) = S(0) \supset S(x) + S(-x)$ for each $x \in A$. Hence if $x \in A$, $S(x) - S(x) \subset (0)$; thus $S(x)$ consists of exactly one element $x^{\#}$ for each $x \in A$. It is clear that the mapping $x \to x^{\#}$ of A into A is an anti-linear injection, and that each F in P is compatible with respect to this mapping. It is obvious that the mapping $x \to x^{\#}$ is the only anti-linear injection on A with respect to which each F in P is compatible. This completes the proof of the theorem.

3. BANACH #-ALGEBRAS.

(3.1) PROPERTIES OF #-ALGEBRAS.

In certain circumstances in the theory of Banach algebras, (for example, when dealing with the second dual of a Banach *-algebra with continuous involution - see (0.1)) one is confronted with an algebra A on which is defined an (anti-linear) involution $x \to x^{\#}$, the latter involution not necessarily satisfying the anti-multiplicative condition $(xy)^{\#} = y^{\#}x^{\#}$ for all $x, y \in A$.

It seems relevant, therefore, to introduce the following definition.

DEFINITION 1. A (Banach) algebra A is said to be a (Banach) #-algebra if there is given an anti-linear involution $x \to x^{\#}$ defined on A.

The study of such algebras is also naturally suggested by the considerations of (2.2).

In this section, we examine some classes of #-algebras, examples of which occur naturally in the theory of Banach algebras. We shall be concerned with #-algebras for which the multiplication is explicitly related to the involution ; each of the relationships considered holds for any *-algebra. (In Chapter 4, we study Banach #-algebras whose involutions satisfy metrical, rather than algebraic, conditions.)

The necessity of imposing constraints on a #-algebra is emphasised by the easily proved fact that any Hamel basis of a lin-

ear space gives rise to an anti-linear involution on the space !
(It would be of interest to know if there exists a continuous anti-linear involution on every Banach space.)

Let A be a #-algebra. $H(A) = \{x \in A : x^{\#} = x\}$ is the set of self-adjoint elements of A. We consider the following possible conditions on A.

(α) For $h \in H(A)$, $h^2 \in H(A)$.

(β) $(x^n)^{\#} = (x^{\#})^n$, for all $x \in A$ and $n \in N$.

(γ) $xx^{\#} \in H(A)$ for all $x \in A$.

(δ) If $h, k \in H(A)$, $i(hk - kh) \in H(A)$.

(ϵ) If $x \in A$, $h \in H(A)$ and $xh = hx$, then $x^{\#}h = hx^{\#}$.

(ζ) $(xy)^{\#} = y^{\#}x^{\#}$, for all $x, y \in A$.

The following theorem obtains certain relationships between these conditions. Part of (2) of the following theorem is contained implicitly in [24].

THEOREM 1. Let A be a #-algebra. Then, on A:

(1) (α) is equivalent to (β).

(2) (γ) and (ζ) are each equivalent to (α) and (δ) together.

(3) (δ) implies (ϵ).

PROOF: (1) Clearly (β) implies (α). Suppose, then, that A satisfies the condition (α). Let $x \in A$ where $x = h + ik$, where h and k belong to $H(A)$. Since $h + k \in H(A)$ and $(h + k)^2 = h^2 + k^2 + (hk + kh)$, we have $hk + kh \in H(A)$. Thus $(x^2)^{\#} = (h^2 - k^2 + i(hk + kh))^{\#} = (x^{\#})^2$.*

Now, if $z, t \in A$, we have $[(z + t)^2]^{\#} = (z^{\#} + t^{\#})^2$, so

* c.f. the latter part of the proof of (1.1), Theorem.

that $(zt + tz)^{\#} = z^{\#}t^{\#} + t^{\#}z^{\#}$.

Suppose that k is a positive integer greater than 1 , and that $x \in A$ such that $(x^{k-1})^{\#} = (x^{\#})^{k-1}$. Then
$(x^k)^{\#} = \frac{1}{2}[x.x^{k-1} + x^{k-1}.x]^{\#} = \frac{1}{2}[x^{\#}.(x^{k-1})^{\#} + (x^{k-1})^{\#}x^{\#}]$
$= (x^{\#})^k$.

An obvious induction arguement now shows that (α) implies (β).

(2) It is clear that if (φ) is true, then (γ) is true. Suppose that (γ) is true. Then (α) is also true. Let $h, k \in H(A)$, and $x = h + ik$. Since $h^2 + k^2 + i(hk - kh) = x^{\#}x \in H(A)$, we have $i(hk - kh) \in H(A)$. Hence (γ) implies both (α) and (δ) .

Finally, suppose that both of the conditions (α) and (δ) hold. Let $h, k \in H(A)$. We show that $(hk)^{\#} = kh$. Let $a = hk$ and $b = kh$. Since $(h + k)^2 \in H(A)$, we have $a + b \in H(A)$, and, by (δ) , we also have $i(a - b) \in H(A)$.

Hence $a + b = a^{\#} + b^{\#}$ and $a - b = b^{\#} - a^{\#}$. Thus $b^{\#} = a$ and $a^{\#} = b$. Hence $(hk)^{\#} = kh$.

Let $x, y \in A$, $x = h + ik$ and $y = h_1 + ik_1$, where $h, h_1, k, k_1 \in H(A)$. Then

$$(xy)^{\#} = (hh_1 - kk_1 + i(hk_1 + kh_1))^{\#} = y^{\#}x^{\#} .$$

Hence (α) and (δ) together imply (φ) , and (2) is proved .

(3) Assume now that (δ) holds. Let $x \in A$ and $h \in H(A)$ for which $xh = hx$. If $x = h_1 + ik_1$, where $h_1, k_1 \in H(A)$, we have $(h_1 + ik_1)h = h(h_1 + ik_1)$, so that $(h_1h - hh_1) = i(hk_1 - k_1h)$. Thus $h_1h - hh_1 \in H(A) \cap iH(A) = (0)$. Hence $h_1h = hh_1$ and $k_1h = hk_1$, so that $x^{\#}h = hx^{\#}$. Thus (δ) implies (ϵ) .

NOTE . It is clear that the condition (δ) is equivalent to the following condition : if h_1, k_1 and $h \in H(A)$ and $(h_1 + ik_1)h = h(h_1 + ik_1)$, then $hh_1 = h_1h$ and $hk_1 = k_1h$.

We now give examples of $\#$-algebras which illustrate the above conditions.

EXAMPLE 1 . Let X be a Banach space on which is defined a continuous anti-linear involution $x \to x^{\#}$. (Many of the most familiar Banach spaces possess natural involutions of this type.) For an operator $T \in B(X)$, define the operator $T^{\#}$ in $B(X)$ by $(T^{\#})(x) = (T(x^{\#}))^{\#}$ $(x \in X)$. It is clear that the mapping $T \to T^{\#}$ defines a continuous anti-linear involution on $B(X)$. It is interesting to note that this involution is actually multiplicative, i.e. if T_1 and T_2 belong to $B(X)$, $(T_1T_2)^{\#} = T_1^{\#} T_2^{\#}$, and hence in particular, satisfies condition (α) . (Banach $\#$-algebras whose involutions satisfy the above multiplicative condition, are, perhaps worthy of study . However, we shall not prosecute this topic in this thesis.)

EXAMPLE 2 . Let A be a Banach $*$-algebra with continuous involution $x \to x^*$. Then, as detailed in (0.1) , there exists a natural anti-linear involution $\mu \to \mu^*$ defined on A'' .

If, in addition, A is commutative, the involution $\mu \to \mu^*$ satisfies condition (α) for each of the Arens multiplications on A'' . (See (0.1), Lemma and Theorem 8.)

We note that if A is any commutative $\#$-algebra, conditions (δ) and (ϵ) are automatically satisfied.

We now introduce the notion of a $(\times, \circ, \#)$-algebra ; formally, the notions of a $(\times, \circ, \#)$-algebra and a $\#$-algebra are identical, but the notations associated with the former are sometimes economical and suggestive.

Let A be a $\#$-algebra and denote the given multiplication in A by "$\times$". For $x, y \in A$, define $x \circ y = (y^{\#} \times x^{\#})^{\#}$. It is easily verified that "$\circ$" defines an associative multiplication on A. When we wish to stress the two multiplications "$\times$" and "$\circ$" on A, we say that A is a $(\times, \circ, \#)$-algebra.

It is clear that the above notation is motivated by the relationship (described in (0.1), Theorem 8) between the involution and the Arens multiplications on the second dual of a Banach $*$-algebra with continuous involution. The situation described in (0.1), Theorem 8 also suggests the following notion. A $(\times, \circ, \#)$-algebra A is said to be a BANACH $(\times, \circ, \#)$-ALGEBRA if there exists a norm $\|\cdot\|$ on A with respect to which A is a Banach algebra under each of the multiplications "$\times$" and "$\circ$".

In general, if A is a $(\times, \circ, \#)$-algebra and A is a Banach algebra with respect to the multiplication "$\times$" for a norm $\|\cdot\|$ on A, then A is also a Banach algebra under the multiplication "$\circ$" for the norm $\|\cdot\|_1$, where, for $x \in A$, $\|x\|_1 = \|x^{\#}\|$.

Let A be a Banach $(\times, \circ, \#)$-algebra. Let $Sp_{\times}(x)$ (resp. $Sp_{\circ}(x)$) and $R_{\times}$ (resp. $R_{\circ}$) denote respectively the spectrum of an element x in A and the radical of A for the multiplication "$\times$" (resp. "$\circ$"). We then have the following simple results. The first three of these results hold for a general $(\times, \circ, \#)$-algebra.

(1) $Sp_{\times}(x) = \overline{Sp_{\circ}(x^{\#})}$ $(x \in A)$.

(2) $(R_{\times})^{\#} = R_{\circ}$.

(3) If A possesses an identity element 1 for the multiplication "$\times$", then $1^{\#}$ is the identity element of A for the multiplication "$\circ$".

(4) If $R_{\times} = (0)$, then the involution $x \to x^{\#}$ is continuous. (This latter fact is a consequence of [13], Theorem 2 .)

We now proceed to examine certain salient features of those Banach $\#$-algebras which satisfy one or more of the conditions detailed at the beginning of this section. We commence by considering a Banach $\#$-algebra A which satisfies condition (α).

LEMMA 1 . Let A be a $\#$-algebra satisfying condition (α). If A possesses an identity element 1, then $1^{\#} = 1$.

PROOF: Suppose that A possesses an identity element 1. By Theorem 1, (1), $(1^{\#})^2 = 1^{\#}$. Also

$$[(1 - 1^{\#})^{\#}]^2 = [(1 - 1^{\#})^2]^{\#}, \text{ so that}$$

$$1^{\#} - 21^{\#} + 1 = 1^{\#} - 21 + 1 \text{ . Thus } 1^{\#} = 1 \text{ .}$$

LEMMA 2 . Let A be a Banach $\#$-algebra which satisfies condition (α), possesses an identity element 1 and has continuous involution. Then A also satisfies the following condition :

if $x \in A$ and $\mathrm{Re}\,\mathrm{Sp}(x) > 0$, then $\mathrm{Sp}(x) \cap -\mathrm{Sp}(x^{\#}) = \phi$.

(We shall refer to this condition as "condition (1)" .)

PROOF: There exists $C \in \mathbb{R}^{++}$ such that $\|x^{\#}\| \leq C\|x\|$ for all $x \in A$, since the involution is continuous. Let $x \in A$. By Theorem 1, (1), $(x^n)^{\#} = (x^{\#})^n$ for each $n \in N$, so that $\|(x^{\#})^n\| = \|(x^n)^{\#}\| \leq C\|x^n\|$ for each $n \in N$. It easily follows that $\rho(x^{\#}) = \rho(x)$.

Now suppose that $\mathrm{Re}\,\mathrm{Sp}(x) > 0$, and let

$U = \{\lambda \in \mathbb{C} : |1 - \lambda| < 1\}$ and $V = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$.

Since U is absorbing in V and $Sp(x)$ is a compact subset of V , there exists $K \in \mathbb{R}^{++}$ such that $(1/K)Sp(x) \subset U$. Thus, using Lemma 1, $\rho(1 - x^{\#}/K) = \rho(1 - x/K) < 1$. Now if there exists $\lambda \in Sp(x) \cap Sp(-x^{\#})$, $\operatorname{Re}\lambda > 0$ and $-\lambda \in Sp(x^{\#})$. But then $\rho(1 - x^{\#}/K) > 1$, and a contradiction would result.

Hence $Sp(x) \cap Sp(-x^{\#}) = \emptyset$.

LEMMA 3 . Let A be a Banach $\#$-algebra with identity element 1 , which satisfies the conditions (α) and (1) . Let $h \in H(A)$ and suppose that $\operatorname{Re} Sp(h) > 0$. Then there exists $x \in H(A)$ such that $x^2 = h$.

PROOF: By Gardner's square-root result (see (0.2), (D), (2)), there exists a unique element $x \in A$ such that $x^2 = h$ and $\operatorname{Re} Sp(x) > 0$. Since, by Theorem 1 , (1), $(x^2)^{\#} = (x^{\#})^2$, we have $(x^{\#})^2 = h$. Let $\lambda \in Sp(x^{\#})$. If $\operatorname{Re}\lambda < 0$, then , since $\operatorname{Re} Sp(x) > 0$ and $x^2 = (x^{\#})^2$, $-\lambda \in Sp(x)$, and condition (1) would be contradicted. Thus $\operatorname{Re} Sp(x^{\#}) > 0$ so that $x = x^{\#}$.

COROLLARY . Let A be a Banach $\#$-algebra which satisfies the condition (α) , has a continuous involution and possesses an identity element 1 . Then if $h \in H(A)$ and $\operatorname{Re} Sp(h) > 0$, there exists $x \in H(A)$ such that $x^2 = h$.

NOTES. (1) The above result generalises Ford's lemma. (See (0.2), (E) .)

(2) It would seem that condition (1) or some similar condition is necessary for Lemma 3 to be valid. Prima facie, one might hope to show that if a Banach $\#$-algebra B satisfies con-

dition (α), then $Sp(x) = \overline{Sp(x^{\#})}$ for all $x \in B$. Unfortunately, the fact, that, from the assumption $xy = yx$ for $x, y \in B$ we can only infer that $(xy)^{\#} = \frac{1}{2}(x^{\#}y^{\#} + y^{\#}x^{\#})$, seems to preclude this possibility.

(3) The above Corollary may also be proved directly using a well-known method. (See, for example, the proof of [20], Lemma 4.1.4.)

THEOREM 2. Let A be a Banach $\#$-algebra with identity 1 which satisfies both (α) and the condition (1) of Lemma 2. Let F be any positive functional on A.* Then, if $h \in H(A)$,

$$|F(h)| \leq \rho(h)F(1) .$$

PROOF: With the help of Lemmas 1 and 3, the result is proved by a standard arguement. (See, for example, [20], p. 214.)

COROLLARY. Let A be a Banach $\#$-algebra with identity element which satisfies (α) and which has continuous involution. Then every positive functional on A is continuous.

We now prove a simple, but important, property of those Banach $\#$-algebras which satisfy condition (ϵ).

THEOREM 3. Let A be a Banach $\#$-algebra satisfying condition (ϵ), and let $h \in H(A)$. Then there exists a closed commutative subalgebra C of A with the properties that $h \in C$, $C^{\#} = C$ and $Sp_C(x) = Sp(x)$ for each $x \in C$.

PROOF: Let $C = \{h\}_{CC}$ and $L = \{x \in A : xh = hx\}$. We show that $C^{\#} = C$. Let $x \in A$ for which $xh = hx$. Since condi-

* See p. 41.

tion (ϵ) is satisfied, $x^{\#}h = hx^{\#}$, so that $L^{\#} = L$. Let $z \in \{h\}_{CC}$ and $k \in L \cap H(A)$. Since $zk = kz$, $z^{\#}k = kz^{\#}$, and, since $L^{\#} = L$, $z^{\#} \in \{h\}_{CC}$. Hence $C^{\#} = C$.

It is well-known that for every $x \in C$, $Sp_C(x) = Sp(x)$.

COROLLARY . The above theorem is also valid if A satisfies condition (ζ) instead of condition (ϵ) .

Let A be a Banach $\#$-algebra ; in accordance with the terminology used in the theory of Banach *-algebras, (see [20], Definition 4.1.6) , we say that the involution in A is HERMITIAN if $Sp(h) \subset \mathbb{R}$ for all $h \in H(A)$.

We conclude this section by examining some results which have been obtained for commutative Banach $\#$-algebras .

We make firstly , some observations which are analogues of facts stated on [20], p. 189 .

Let A be a commutative Banach $\#$-algebra . In the $(\times, \circ, \#)$-notation introduced earlier , A is a Banach algebra for each of the multiplications "$\times$" and "$\circ$" . (See p. 48 .) Let $\Phi_\times$ and $\Phi_\circ$ denote the carrier spaces for the multiplications "$\times$" and "$\circ$" .

Let $\phi \in \Phi_\times$. We may define the linear functional $\phi^{\#}$ on A by $(\phi^{\#})(x) = \phi(x^{\#})$ $(x \in A)$. It is clear that $\phi^{\#} \in \Phi_\circ$. We define the mapping I of $\Phi_\times$ into $\Phi_\circ$ by $I\phi = \phi^{\#}$, where $\phi \in \Phi_\times$. It is easily verified that I is a homeomorphism of $\Phi_\times$ onto $\Phi_\circ$ and that $I(\partial_A \Phi_\times) = \partial_A \Phi_\circ$, $\partial_A \Phi_\times$ (resp. $\partial_A \Phi_\circ$) being the A-boundary of $\Phi_\times$ (resp. $\Phi_\circ$) . (See [20], p. 142 .)

We now discuss some properties of those commutative Banach $\#$-algebras which have hermitian involutions .

THEOREM 4 . Let A be a commutative Banach $\#$-algebra with hermitian involution and radical R . Then for all $x, y \in A$,

$$(xy)^{\#} - y^{\#} x^{\#} \in R \ .$$

PROOF: Let $x \in A$ and $\phi \in \Phi_A$. Since the involution is hermitian, $\widehat{(x^{\#})}(\phi) = \overline{\hat{x}(\phi)}$. Hence, if $y \in A$,

$$\widehat{((xy)^{\#} - y^{\#} x^{\#})}(\phi) = \overline{\hat{x}(\phi)\hat{y}(\phi)} - \overline{\hat{x}(\phi)\hat{y}(\phi)} = 0 \ .$$

The required result follows immediately .

NOTE . It is easy to see that in the situation of Theorem 4 , $R^{\#} = R$.

We now give a necessary and sufficient condition for the involution on a commutative Banach $\#$-algebra with identity element to be hermitian . This result is an analogue of [20], Lemma 4.2.1.

THEOREM 5 . Let A be a commutative Banach $\#$-algebra with identity element 1 and radical R . Then $x \to x^{\#}$ is a hermitian involution on A if and only if $1^{\#} - 1 \in R$ and $(\rho(x))^2 = \rho(xx^{\#})$ for all $x \in A$.

PROOF: Suppose firstly that the involution $x \to x^{\#}$ is hermitian . If $\phi \in \Phi_A$, then $\widehat{(1^{\#})}(\phi) = \overline{\hat{1}(\phi)} = 1$, so that $1^{\#} - 1 \in R$. It is also clear that if $x \in A$ we have $\widehat{(xx^{\#})}(\phi) = |\hat{x}(\phi)|^2$, so that $(\rho(x))^2 = \rho(xx^{\#})$ for all $x \in A$. This proves one implication of the theorem .

Now suppose that A satisfies the two conditions specified above and that the involution on A is not hermitian . Then there exists $h \in H(A)$ and $\lambda \in Sp(h)$ such that $\lambda = a + ib$, where $a, b \in \mathbb{R}$ and $b \neq 0$. Let $u = (1/b)(h - a1)$. Then $u^{\#} - u \in R$, and there exists $\phi \in \Phi_A$ such that $\hat{u}(\phi) = i$. For each $n \in N$, let $x_n = (u + in1)$. Let $n \in N$. Then

$$(1 + n)^2 \leq (\rho(x_n))^2 = \rho(x_n x_n^{\#}) \quad .$$

$$\text{Now } x_n x_n^{\#} = (u + in1)(u^{\#} - in1^{\#})$$
$$= uu^{\#} + ni(u^{\#} - u1^{\#}) + n^2 1^{\#} \quad .$$

Since $u^{\#} - u$ and $1^{\#} - 1$ belong to R, and as $\rho(1^{\#}) = 1$, we have, for each $n \in N$, $(1 + n)^2 \leq \rho(uu^{\#}) + n^2$. As this is obviously impossible for all $n \in N$, the theorem is proved.

NOTE. It would be of interest to know if an analogue of [20], Lemma 4.2.1 can be obtained for commutative Banach $\#$-algebras not necessarily containing identity elements.

We now ask the following question : when does a commutative Banach $\#$-algebra A with identity element have the property that, if M is any maximal ideal of A, $M^{\#}$ is also a maximal ideal of A ?

The next theorem answers this question.

THEOREM 6. Let A be a commutative Banach $\#$-algebra with identity element 1. Then $M^{\#}$ is a maximal ideal of A for every maximal ideal M of A if and only if A satisfies the following condition :

$$x \in A \text{ and } 0 \in Sp(x) \text{ implies } 0 \in Sp(x^{\#}) \quad .$$

PROOF: It is easy to show that if $M^{\#}$ is a maximal ideal in A for every maximal ideal M of A, then the above condition is satisfied.

Conversely, suppose that the above condition is satisfied and that M is a maximal ideal in A. Then $M^{\#}$ is a subspace of codimension 1 in A and further, by the above condition, $M^{\#}$ consists of singular elements of A. Hence, by (0.1), Theorem 7, $M^{\#}$ is a maximal ideal in A and the result is proved.

COROLLARY 1. Suppose that A is a semi-simple commutative Banach $\#$-algebra with identity 1 and that A satisfies the condition detailed in the statement of Theorem 6. Then, for all $x, y \in A$, $x^{\#}y^{\#} = (xy(1^{\#})^{-1})^{\#}$.

PROOF: (It is easily verified from Theorem 6 that $1^{\#}$ is a regular element of A, so that the required result is meaningful.)

Let $\phi \in \Phi_A$. Then there exists (by Theorem 6) $\phi_{\#} \in \Phi_A$ such that $N_{(\phi_{\#})} = (N_{\phi})^{\#}$.

Let $x \in A$. Since $x - \hat{x}(\phi)1 \in N_{\phi}$, we have

$$\phi_{\#}(x^{\#} - \overline{\hat{x}(\phi)}\,1^{\#}) = 0, \text{ i.e. } \widehat{(x^{\#})}(\phi_{\#}) = \overline{\hat{x}(\phi)}\,\widehat{(1^{\#})}(\phi_{\#}).$$

In particular, $1 = \overline{\widehat{(1^{\#})}(\phi)}\,\widehat{(1^{\#})}(\phi_{\#})$, so that

$\widehat{(1^{\#})}(\phi_{\#}) = \overline{\widehat{((1^{\#})^{-1})}(\phi)}$. Thus $\widehat{(x^{\#})}(\phi_{\#}) = \overline{\hat{x}(\phi)}\,\overline{\widehat{(1^{\#})^{-1}}(\phi)}$.

If $y \in A$, we have

$$\begin{aligned}\widehat{(x^{\#}y^{\#})^{\#}}(\phi) &= \overline{\widehat{(x^{\#}y^{\#})}(\phi_{\#})}\,\overline{\widehat{((1^{\#})^{-1})}(\phi_{\#})} \\ &= \overline{\widehat{(x^{\#})}(\phi_{\#})}\,\overline{\widehat{(y^{\#})}(\phi_{\#})}\,\overline{\widehat{(1^{\#})^{-1}}(\phi_{\#})} \\ &= \hat{x}(\phi)\hat{y}(\phi)\widehat{(1^{\#})^{-1}}(\phi).\end{aligned}$$

Hence, for all $\psi \in \Phi_A$, $\widehat{((x^{\#}y^{\#})^{\#} - xy(1^{\#})^{-1})}(\psi) = 0$, and the required result follows.

COROLLARY 2. Let A be a semi-simple commutative Banach $\#$-algebra with identity 1. Then A is a Banach $*$-algebra under the given involution $x \to x^{\#}$ if and only if, for each $x \in A$,

$Sp(x) = \overline{Sp(x^{\#})}$.

Finally we give a generalisation of Corollary 2 .

THEOREM 7 . Let A be a Banach $\#$-algebra satisfying the following conditions :

(1) For $h, k \in H(A)$, $i(hk - kh) \in H(A)$.

(2) $H(A) \setminus \{0\}$ has no topologically nilpotent elements .

(3) For each $x \in A$, $Sp(x) = \overline{Sp(x^{\#})}$.

Then A is a Banach $*$-algebra under the involution $x \to x^{\#}$.

PROOF: Suppose firstly that A possesses an identity element 1 . Since $Sp(1^{\#}) = \{1\}$, we have, using condition (2) of the theorem , $1^{\#} = 1$.

(If A has no identity element , we adjoin an identity element 1 ; $B = A + \mathbb{C}1$ is a Banach algebra and we may extend the involution on A to one on B by defining for $\lambda \in \mathbb{C}$ and $a \in A$, $(\lambda 1 + a)^{\#} = \bar{\lambda}1 + a^{\#}$. It is easy to verify that B as a Banach $\#$-algebra under the above involution satisfies all of the conditions (1) , (2) and (3) of the theorem . These observations show that we need only consider the case when A possesses an identity element .)

Let $h \in H(A)$. By Theorem 3 , Corollary , there exists a closed commutative subalgebra C of A with the properties that both 1 and h belong to C , $C^{\#} = C$ and $Sp_C(x) = Sp(x)$ for all $x \in C$. It is easily verified (using Theorem 6) that, if R is the radical of C , $R^{\#} = R$; using condition (2) , it follows that C is semi-simple . Hence by Theorem 6 , Corollary 2 , C is a Banach $*$-algebra under the involution $x \to x^{\#}$ ($x \in C$) , so that $h^2 \in H(A)$. An application of Theorem 1 , (2) now yields the required result .

(3.2) REPRESENTATIONS AND POSITIVE FUNCTIONALS ON BANACH #-ALGEBRAS .

Throughout this section , A will denote a Banach #-algebra .

Let F be an admissable positive functional on the Banach #-algebra A . (See (2.2) .) It is clear that the representation of A on Hilbert space associated with F will normally possess stronger properties than a representation of A associated with an arbitrary admissable generalised positive functional on A . It is natural to ask : when is a representation of A on Hilbert space associated with an admissable positive functional on A ?

We introduce the following notation : if $R : a \to T_a$ is a representation of A on the Hilbert space H , R^* is the corresponding anti-representation $a \to (T_a)^*$. Also

$P_A(R) = \{(x,y) : x,y \in H \text{ and } (T_{a^\# a}x,y) \geq 0 \text{ for all } a \in A\}$.

The following theorem gives a partial answer to the above question . We use notations employed in (2.2) .

THEOREM 1 . Let A be a Banach #-algebra and $R : a \to T_a$ be a representation of A on the Hilbert space H . Suppose that x,y are vectors in H , which are topologically cyclic for R and R^* respectively and such that $(x,y) \in P_A(R)$. Then there exists an admissable positive functional F on A such that the representation $a \to \hat{T}^F_a$ associated with F is isometrically equivalent to R .

Conversely , if A contains an identity element , F is an admissable positive functional on A and R^F denotes the representation $a \to T_a^F$ of A on X_F^- , there exists $(x,y) \in P_A(R^F)$ such that x and y are topologically cyclic vectors for the mappings R^F and $(R^F)^*$ respectively .

The proof of the above theorem is standard* and is omitted .

It is clear that if $R : a \to T_a$ is a representation of the Banach #-algebra A on a Hilbert space H , and $(x,y) \in P_A(R)$, then, for all $a,b \in A$,

$$(T_{b^{\#}a}x,y) = \overline{(T_{a^{\#}b}x,y)} \quad .$$

It would appear that few of the results which have been obtained for positive functionals on Banach *-algebras admit direct generalisation within the general context of Banach #-algebras .

To illustrate this , we consider the natural generalisation to the context of Banach #-algebras of the notion of a representable positive functional on a Banach *-algebra . (See [20], p. 216 .)

DEFINITION 1 . A positive functional F on A is said to be REPRESENTABLE by a representation $R : a \to T_a$ of A on a Hilbert space H if there exist topologically cyclic vectors x and y in H for the mappings R and R^* such that $F(a) = (T_a x,y)$ for all $a \in A$.

We might enquire whether an analogue of [20] , Lemma 4.5.6 holds in this situation , i.e. given a representation $R : a \to T_a$ of A on a Hilbert space H and given $(x,y) \in P_A(R)$, does there exist a (closed) invariant subspace H_0 of H such that

* c.f. (2.2) , Theorems 1 and 4 .

the positive functional F on A, given by $F(a) = (T_a x, y)$ $(a \in A)$ is representable by the restriction of R to H_0 ? The following example shows that this is not the case in general, even when $x = y$.

EXAMPLE 1. Let $H = \{(x,y,z)' : x,y,z \in \mathbb{C}\}$ and $A = \left\{ \begin{bmatrix} \cdot & \cdot & \cdot \\ \lambda & \cdot & \cdot \\ \mu & \lambda & \cdot \end{bmatrix} : \lambda, \mu \in \mathbb{C} \right\}$. For $T = \begin{bmatrix} \cdot & \cdot & \cdot \\ \lambda & \cdot & \cdot \\ \mu & \lambda & \cdot \end{bmatrix}$, where $\lambda, \mu \in \mathbb{C}$, let $T^{\#} = \begin{bmatrix} \cdot & \cdot & \cdot \\ \bar{\lambda} & \cdot & \cdot \\ \bar{\mu} & \bar{\lambda} & \cdot \end{bmatrix}$. Then A is a Banach $\#$-algebra under the involution $T \to T^{\#}$. The identity mapping I on A is obviously a representation of A on H. Let $u = (1,1,1)'$ and let $F(T) = (Tu, u)$ for all $T \in A$. Then F is a positive functional on A. Since every element of A is nilpotent, there is no proper invariant subspace of H containing a cyclic vector; since $F \neq 0$, F is thus not representable by the restriction of I to any invariant subspace of H.

The above considerations make it seem, therefore, desirable to consider positive functionals on the Banach $\#$-algebra A which satisfy certain restrictive conditions. From this point of view, the following Lemma is suggestive.

LEMMA 1. Let A be a Banach $\#$-algebra with identity and $R : a \to T_a$ be a representation of A on the Hilbert space H. Then $(x,x) \in P_A(R)$ for all $x \in H$ if and only if $(T_a)^* = T_{a^{\#}}$ for all $a \in A$.

PROOF: Suppose that $(x,x) \in P_A(R)$ for all $x \in H$. Let $H^1 = \{T \in B(H) : T^* = T\}$. As in (3.1), $H(A) = \{a \in A : a^{\#} = a\}$.

We note that for each $a \in A$, $T_{a^{\#}a} \in H^1$. Let $1 = h + ik$

be the identity element of A, where $h, k \in H(A)$. Since $T_{1^{\#}} = T_{1^{\#}1}$, $T_{1^{\#}} \in H^1$. Now, for each $x \in H$,
$(T_1 x, x) = (T_1 T_1 x, x) = \overline{(T_{1^{\#}} T_{1^{\#}} x, x)} \geq 0$; hence $T_1 \in H^1$. It follows that T_h, $T_{ik} \in H^1$. As $T_{ik} \in H^1$,
$-(T_{(k^2)} x, x) = ((T_{ik})^2 x, x) \geq 0$ for all $x \in H$. But since $k \in H(A)$, $(T_{(k^2)} x, x) \geq 0$ for all $x \in H$, and hence $(T_{(k^2)} x, x) = 0$ for all $x \in H$. Hence $(T_k)^2 = 0$, so that, since $T_{ik} \in H^1$, $T_k = 0$. Thus $T_1 = T_h$.

Finally, let $a \in A$ and $x \in H$. Then
$(T_a x, x) = (T_h T_a x, x) = \overline{(T_{a^{\#}} T_h x, x)} = (x, T_{a^{\#}} x)$.

Hence $((T_{a^{\#}} - (T_a)^*) x, x) = 0$ for all $x \in H$, so that $T_{a^{\#}} = (T_a)^*$. This proves one implication of the lemma. As the other implication is obvious, the result follows.

If $R : a \to T_a$ is a representation of a Banach $\#$-algebra A on a Hilbert space, R is said to be a *-REPRESENTATION if $T_{a^{\#}} = (T_a)^*$ for all $a \in A$.

Let F be an admissable positive functional on the Banach $\#$-algebra A. Examination of the above lemma and the conditions which must be satisfied by F in order that the representation $a \to \hat{T}^F_a$ on $\bar{X}_F$ should be a *-representation indicates that the following definition may be relevant.

DEFINITION 2. A positive functional F on A is said to be STRONGLY POSITIVE if F_u is a positive functional for all $u \in A$.*

We introduce the following notation :

$$M_2 = \left\{ (x_1 x_2)^{\#} - x_2^{\#} x_1^{\#} : x_1, x_2 \in A \right\} .$$

* For the notation F_u, see [20], p. 214.

LEMMA 2 . Let F be a positive functional on A . Then

$$P_F = \bigcap_{u \in A} N_{F_u}$$

PROOF: It is easy to see that $P_F \subset \bigcap_{u \in A} N_{F_u}$. Let $x \in \bigcap_{u \in A} N_{F_u}$. For $h, k \in H(A)$, we have

$$F[(h - ik)x(h + ik)] = 0 = F[(h + k)x(h + k)] .$$

Hence $F(hxk) = 0$. It easily follows that $x \in P_F$ so that the result is proved .

LEMMA 3 . Suppose that A possesses an identity element 1 for which $1^{\#} = 1$. Let F be a strongly positive functional on A . Then A/P_F is a *-algebra .

PROOF: We note that if G is any positive functional on A , $(N_G)^{\#} = N_G$. (For, using (2.2) , Lemma 1 , $G(x^{\#}) = \overline{G(x)}$ for all $x \in A$.)

Hence , for each $u \in A$, $(N_{F_u})^{\#} = N_{F_u}$, and so , by Lemma 2 , $(P_F)^{\#} = P_F$.

We now maintain that $M_2 \subset P_F$. This is an easy consequence of Lemma 1 and the fact that $F_u(xy) = F_u((y^{\#} x^{\#})^{\#})$ for all $u, x, y \in A$.

It is immediate that A/P_F is a *-algebra under the involution $(x + P_F) \rightarrow x^{\#} + P_F$ $(x \in A)$.

COROLLARY . For each $u \in A$, F_u is a strongly positive functional on A .

We require the following lemma .

LEMMA 4 . Let B be a Banach algebra with identity 1 and P

be a two-sided ideal in B . Let $B^1 = B/P$ and $B^2 = B/P^-$. For $x \in B$, let $x^1 = x + P$ and $x^2 = x + P^-$. Then , for all $x \in B$, $|Sp_{B^1}(x^1)| = |Sp_{B^2}(x^2)|$.

PROOF: Let π be the canonical mapping of B onto B^1 . If M_1 is a maximal left ideal in B^1, $M = \pi^{-1}(M_1)$ is closed in B . Hence $P^- \subset M$ and thus $\pi P^- \subset M_1$. Hence , πP^- is contained in the radical of B^1 . Since $B^1/\pi P^-$ may be identified with B^2 in a natural way , the required result follows easily .

THEOREM 1 . Let A be a Banach $\#$-algebra with a self-adjoint identity element 1 , and suppose that the involution in A is continuous and that A satisfies condition (ϵ) of (3.1) . (See p. 45 .) Let F be a strongly positive functional on A . Then if $a \in H(A)$ and $\rho(a) < 1$, there exists $h \in H(A)$ such that

$$F(h \circ h - a) = 0 .$$

PROOF: Let $a \in H(A)$ such that $\rho(a) < 1$. By a result due to Bonsall (see (0.2) , (A)) , there exists $x \in A$ such that $x \circ x = a$, where by (0.2) , (D) , (1) and (0.2) , Theorem 3 ,

$$\{x\}_{CC} = \{a\}_{CC} .$$

Now , $xa = ax$, and hence by condition (ϵ) , $x^{\#}a = ax^{\#}$. Thus $xx^{\#} = x^{\#}x$.

Let $A^1 = A/P_F$ and $A^2 = A/P_F^-$. For $y \in A$ let $y^1 = y + P_F$ and $y^2 = y + P_F^-$. Using Lemma 3 and the continuity of the involution in A , A^1 and A^2 are *-algebras under the involutions $x^1 \to (x^{\#})^1$ and $x^2 \to (x^{\#})^2$ respectively $(x \in A)$. (A^2 is clearly a Banach *-algebra .)

Using Lemma 4 , we have

$$|Sp_{A^1}(x^1 + (x^{\#})^1)| = |Sp_{A^2}(x^2 + (x^{\#})^2)|$$
$$\leq \rho_{A^2}(x^2) + \rho_{A^2}((x^{\#})^2)$$
$$= 2\rho_{A^2}(x^2) \leq 2\rho_A(x) < 2 \ .$$

Since $x^1 \circ x^1 = a^1 = (x^{\#})^1 \circ (x^{\#})^1$, we have

$$(x^1 - (x^{\#})^1)(2 - x^1 - (x^{\#})^1) = 0^1 \ .^*$$

Since , by the above arguement , $|Sp_A 1(x^1 + (x^{\#})^1)| < 2$, $21^1 - x^1 - (x^{\#})^1$ is invertible in A^1 , and therefore $x^1 = (x^{\#})^1$. It is now easy to show that if $h = \frac{1}{2}(x + x^{\#})$, $F(h \circ h - a) = 0$.

With the help of Lemma 3 , Corollary and Theorem 1 , it is not difficult to prove the following result . (c.f. [20], Theorem 4.5.2 .)

THEOREM 2 . Let A be a Banach $\#$-algebra satisfying the conditions of Theorem 1 . Then , if F is a strongly positive functional on A , $|F_u(h)| \leq \rho(h)F(u^{\#}u)$, for $u \in A$ and $h \in H(A)$.

COROLLARY . F is continuous on A , and is an admissable positive functional on A .

The following theorem now follows immediately .

THEOREM 3 . Let A be a Banach $\#$-algebra satisfying the conditions of Theorem 1 . Then if F is a strongly positive functional on A , the representation $a \to \hat{T}^F_a$ on $\bar{X}_F$ is defined , and is a topologically cyclic $*$-representation .

Conversely , if $a \to T_a$ is a topologically cyclic $*$-representation of A on Hilbert space , the representation $a \to T_a$ is

* This part of the proof was suggested by part of Professor F. F. Bonsall's original proof of his square-root result contained in (0.2) , (A) .

unitarily equivalent to the representation $a \to \hat{T}^F_a$ on $\bar{X}_F$,
where F is a strongly positive functional on A .

4. METRICAL CHARACTERISATIONS OF B*-ALGEBRAS .

(4.1) ASPECTS OF THE VIDAV-PALMER THEOREM *.

The Vidav-Palmer theorem (see [24] and [18]) is perhaps one of the most outstanding results which has been proved in the theory of numerical range . In this section , we discuss certain aspects of this theorem , laying particular emphasis on those facets of the result which may be illumined by simple Functional analytic techniques as distinct from the powerful theorems of complex analysis originally evoked by Vidav .

To the author's knowledge , no proof of the above theorem which does not involve results from complex analysis has yet been given . (We may compare this situation to similar situations in the proofs of [20], Theorem 3.6.3 and (0.1) , Theorem 7 .) However, the results proved in this section are sufficient for the purposes of (4.2) .

We commence by defining the notion of a "Vidav algebra" .

DEFINITION 1 . A unital Banach algebra A is said to be a VIDAV ALGEBRA if there exists a subset $H(A)$ of A with the following properties :

(1) $H(A) + iH(A) = A$.

(2) For each $h \in H(A)$, $\|1 + i\alpha h\| \leq 1 + o(\alpha)$ $(\alpha \in \mathbb{R}, \alpha \to 0)$.

NOTE. In [24], Vidav studied an algebra A which satisfied a third condition in addition to the conditions (1) and (2) of Definition 1 . However , in [18], Palmer has shown that this third

* Some of the results of this section have also been independently observed by Bonsall and Duncan in [6] .

condition is unnecessary .

The Vidav-Palmer theorem (see [6], Theorem 6.9) states that every Vidav algebra is isometrically *-isomorphic with a C*-algebra .

In the sequel , A will denote a Vidav algebra unless otherwise explicitly stated ; $H(A)$ will be the given set of elements of A satisfying the conditions (1) and (2) of Definition 1 .

We have the following simple theorem . (See [6], Lemmas 5.7 and 5.8 .)

THEOREM 1 . Let A be a Vidav algebra . Then :

(1) $H(A)$ is the set of hermitian elements of A .

(2) For each $x \in A$, there exist unique elements $h, k \in H(A)$ such that $x = h + ik$.

(3) For $x \in A$ where $x = h + ik$, h and k belonging to $H(A)$, let $x^* = h - ik$. Then the mapping $x \to x^*$ is a continuous (anti-linear) involution on A .

PROOF: Using condition (2) of Definition 1 and (0.1) , Theorem 4, (3) , it is clear that $H(A)$ consists of hermitian elements of of A .

Now suppose h, k, h' and k' are hermitian elements of A , and that $h + ik = h' + ik'$. Then $V(h - h') = V(k - k') = (0)$, so that $h = h'$ and $k = k'$ by (0.1), Theorem 4, (2) .
The first two parts of the theorem follow easily from the above observations .

We now prove (3) of the theorem . Let $\{x_n\}$ be a sequence in A such that $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} (x_n)^* = s$, where

$s \in A$. For each $n \in N$, let $x_n = h_n + ik_n$, where $h_n, k_n \in H(A)$. Let $F \in D(1)$. Then $\lim_{n\to\infty} [F(h_n) + iF(k_n)] = 0$, so that $\lim_{n\to\infty} F(h_n) = \lim_{n\to\infty} F(k_n) = 0$. Hence $F(s) = 0$. It follows that $V(s) = (0)$. Thus by (0.1), Theorem 4, (2), $s = 0$. An application of the closed-graph theorem now yields the required result.

THEOREM 2. Let B be a $*$-algebra with identity 1. Then B is a B^*-algebra with respect to the given involution on B if and only if there exists a Banach algebra norm $\|\cdot\|$ on B making B into a unital Banach algebra such that $D_{\|\cdot\|}(1)$ consists of positive functionals.

PROOF: If B is a B^*-algebra under a norm $\|\cdot\|$, every element of $D_{\|\cdot\|}(1)$ is a positive functional, (See [20], Theorem 4.8.16.)

Suppose now that B is a unital Banach $*$-algebra under a norm $\|\cdot\|$ such that $D_{\|\cdot\|}(1)$ consists of positive functionals.

It is easy to verify directly that if we define for each $x \in A$, $|x|^2 = \sup \{F(x^*x) : F \in D(1)\}$, $|\cdot|$ is a pseudo-norm on A such that $|x^*x| = |x|^2$. (See [20], Corollary 4.6.10.)

By (0.1), Theorem 4, (2) and [20], Theorem 4.1.15, (or, alternatively, Theorem 1, (3)), there exists $C \in \mathbb{R}^{++}$ such that $\|x^*\| \leq C\|x\|$ for all $x \in A$.

We then have for each $x \in A$,

$$|x|^2 = v(x^*x) \leq \|x^*x\| \leq C\|x\|^2 \leq Ce[v(x)]^2 \leq Ce|x|^2 ,$$

using (0.1), Theorem 1 and the fact that $|F(x)|^2 \leq F(x^*x)$ for all $F \in D(1)$. Hence, B is a B^*-algebra under the norm $|\cdot|$.

NOTE . In actual fact, of course, B of Theorem 2 is a B*-algebra under $\|\cdot\|$. (This follows from the Vidav-Palmer theorem .)

LEMMA 1 . Let ϕ be a mapping of $\mathbb{R}^{++}$ into $\mathbb{R}$ for which $\lim_{\alpha\to 0^+} \phi(\alpha)/\alpha \leq 0$ and $\phi(n\alpha) \leq n\phi(\alpha)$, for all $\alpha\in\mathbb{R}^{++}$ and $n\in N$. Then $\phi(\alpha) \leq 0$ for all $\alpha\in\mathbb{R}^{++}$.

PROOF: Let $\lambda\in\mathbb{R}^{++}$ and $\epsilon > 0$. Then there exists $\alpha_0\in\mathbb{R}^{++}$ such that for all $\alpha\in\mathbb{R}^{++}$ for which $\alpha\leq\alpha_0$, $\phi(\alpha)/\alpha < \epsilon$. Choose $\alpha'\in\mathbb{R}^{++}$ and $n\in N$ such that $0<\alpha'\leq\alpha_0$ and $n\alpha' = \lambda$. Then $\phi(\lambda)/\lambda \leq n\phi(\alpha')/n\alpha' < \epsilon$, i.e. $\phi(\lambda) < \epsilon\lambda$. Since ϵ was an arbitrary positive number, $\phi(\lambda) \leq 0$, and the required result follows .

COROLLARY . If $\lim_{\alpha\to\infty} \phi(\alpha)/\alpha \geq 0$, then $\phi = 0$.

PROOF: Let $\lambda\in\mathbb{R}^{++}$. Then, for all $n\in N$, $\phi(n\lambda)/(n\lambda) \leq \phi(\lambda)/\lambda$. If $\lim_{\alpha\to\infty} \phi(\alpha)/\alpha \geq 0$, we see that $\phi(\lambda) \geq 0$. Combining this result with Lemma 1 , we see that $\phi = 0$.

LEMMA 2 . Let B be a unital Banach algebra . For $x\in B$, let $\delta(x) = \lim_{\alpha\to 0^+} \alpha^{-1}\{\|1+\alpha x\| - 1\}$. Then $\|e^{\alpha x}\| = e^{\alpha\delta(x)}$ for all $\alpha\in\mathbb{R}^{++}$ if and only if $\sup \mathrm{Re}\, \mathrm{Sp}(x) = \delta(x)$.

PROOF: Let $x\in B$ for which $\sup \mathrm{Re}\,\mathrm{Sp}(x) = \delta(x)$. For $\alpha\in\mathbb{R}$ let $\phi(\alpha) = \log\|e^{\alpha x}\| - \alpha\delta(x)$.

It is clear that if $\alpha\in\mathbb{R}$, $\phi(n\alpha) \leq n\phi(\alpha)$ for all $n\in N$. Further

$$\lim_{\alpha\to 0^+} \phi(\alpha)/\alpha = \lim_{\alpha\to 0^+} \alpha^{-1}\{\|1+\alpha x\| - 1\} - \delta(x) = 0 .$$

Now, $$\lim_{n\to\infty} \phi(n)/n = \lim_{n\to\infty} \log(\|e^{nx}\|^{1/n}) - \delta(x)$$

$$= \log [\rho(e^x)] - \delta(x) = 0 .$$

Hence , by the corollary to Lemma 1 , $\phi(\alpha) = 0$ for all $\alpha \in \mathbb{R}^{++}$, and the first part of the result is proved .

Suppose now that $x \in B$ and that $\| e^{\alpha x} \| = e^{\alpha \delta(x)}$ for all $\alpha \in \mathbb{R}^{++}$. Then

$$\sup \operatorname{Re} Sp(x) = \lim_{n \to \infty} n^{-1} \log \| e^{nx} \| = \lim_{n \to \infty} n^{-1} (n \delta(x)) = \delta(x) .$$

This concludes the proof of the lemma .

For the following theorem , see [24], Lemmas 1 and 2 .

THEOREM 3 . Let A be a Vidav algebra . Then if $h, k \in H(A)$ $\| e^{ih} \| = 1$ and $i(hk - kh) \in H(A)$. (Thus A satisfies condition (δ) of (3.1) for the involution $x \to x^*$ on A mentioned in Theorem 1 , (3) .)

PROOF: Let $h \in H(A)$. By (0.1), Theorem 4, (1),

$$\operatorname{Re} Sp(ih) \subset \operatorname{Re} V(ih) = (0) .$$

Hence $\sup \operatorname{Re} Sp(ih) = 0 = \delta(ih)$. Thus, by Lemma 2 , $\| e^{ih} \| = 1$.

The fact that , if $h, k \in H(A)$, $i(hk - kh) \in H(A)$, follows easily as in [24], Lemma 2, (b) .

We now note two easily verified properties of Vidav algebras. In the sequel , a Vidav algebra A will be assumed to be a Banach #-algebra with involution $x \to x^*$ mentioned in Theorem 1, (3) .

THEOREM 4 . Let A be a Vidav algebra . Then :

(1) If $h \in H(A)$ and $F \in D(1)$, $\operatorname{Re} F(h^2) \geq 0$.

(2) If I is a closed two-sided ideal in A , then $I^* = I$.

PROOF: (1) Let $h \in H(A)$ and $F \in D(1)$. Then, for $\alpha \in \mathbb{R}$,

$\mathrm{Re}\, F(e^{i\alpha h}) \le \| e^{i\alpha h} \| = 1$, using Theorem 3 . Hence $1 - \alpha^2/2\ \mathrm{Re}F(h^2) \le 1 + o(\alpha^3)$ $(\alpha \in \mathbb{R}, \alpha \to 0)$. Thus $-\alpha^2/2\ \mathrm{Re}F(h^2) \le o(\alpha^3)$. This is possible if and only if $\mathrm{Re}F(h^2) \ge 0$.

(2) Let I be a closed two-sided ideal in A and let $A^1 = A/I$. For $x \in A$, let $x^1 = x + I$. Let $H(A^1) = \{h^1 : h \in H(A)\}$. Then $A^1 = H(A^1) + iH(A^1)$. Further, if $\|\cdot\|^1$ denotes the norm induced on A^1 by the norm $\|\cdot\|$ of A and $h \in H(A)$, we have $\| 1^1 + i\alpha h^1 \|^1 \le \| 1 + i\alpha h \| \le 1 + o(\alpha)$ for $\alpha \in \mathbb{R}$ and $\alpha \to 0$. Hence A^1 is a Vidav algebra, and using Theorem 1 , (2) ,it follows that $I^* = I$.

NOTE . The proof of (2) of the above theorem is essentially due to F. F. Bonsall , who used it to prove that every closed two-sided ideal of a B*-algebra is a *-ideal . (See [6], Theorem 7.7 .)

THEOREM 5 . Let A be a Vidav algebra which contains no non-zero topologically nilpotent hermitian elements . Then A is a B*-algebra .

PROOF: Let $h \in H(A)$. Using (3.1), Theorem 3, Corollary, $B = \{h\}_{CC}$ is a *-subalgebra of A , where we are using obvious terminology .

Since B is a Vidav algebra in its own right, B has hermitian involution (by (0.1), Theorem 4, (1)) and hence, by (3.1), Theorem 4, $x^*y^* - (xy)^* \in R$, where $x, y \in B$ and R is the radical of B . Since $R^* = R$, (see (3.1), Theorem 4, Note) , and as 0 is the only topologically nilpotent element in A , $R = (0)$. Hence B is a Banach *-algebra .

Hence , $h^2 \in H(A)$, and so, by Theorem 4, (1) , $F(h^2) \geq 0$ for every $F \in D(1)$. Hence , if $x \in B$ and $x = h + ik$, where $h, k \in H(A)$, $F(x^*x) = F(h^2 + k^2) \geq 0$ for every $F \in D(1)$. Hence, by Theorem 2 , B is a commutative B*-algebra .

Further, since A satisfies each of the conditions (α) and (δ) of (3.1), A is a Banach *-algebra by (3.1), Theorem 1, (2) .

We now show that A is a B*-algebra .

Let $h \in H(A)$. Then $\{h\}_{CC}$ is a B*-algebra under the norm $\|\cdot\|_0$ where for $x \in \{h\}_{CC}$, $\|x\|_0^2 = \sup\{F(x^*x) : F \in D(1)\}$. Hence $\rho(h) = \rho_C(h) = \|h\|_0 \geq v(h)$. Hence $\rho(h) = v(h)$.

It is now easy to show that if $h, k \in H(A)$,

$$\rho(h + k) \leq \rho(h) + \rho(k) .$$

If $x = h + ik$, where $h, k \in H(A)$, and $x^*x = xx^* = 0$, then $h^2 + k^2 = 0$. Hence $h^2 = k^2 = 0$, so that $h = k = 0$, and $x = 0$.

Using these facts and the methods of [20], Lemma 4.8.8 and Theorem 4.8.9 , it is not difficult to show that A is symmetric.* Since every element of D(1) is weakly positive on A , each such element is a positive functional on A . Hence, by Theorem 2 , A is a B*-algebra .

It is clear that the establishment of the Vidav-Palmer theorem within the context of Theorem 5 depends crucially on showing that every (commutative) Vidav algebra is semi-simple . It would appear that the proof of this fact requires an application of "Vidav's lemma" . (See [24], Lemma 3 and [6], §5 .) We require only part of Vidav's lemma . The proof is omitted . (All

* Alternatively , one could use a deep result of Shirali and Ford . (See [22] .)

proofs of this lemma known to the author depend on complex analysis ; the simplest proof of the lemma seems to be one due to Lumer. This proof is given in [6], §5 .)

THEOREM 6 . Let A be a Vidav algebra, and suppose $h \in H(A)$ such that $\rho(h) \in Sp(h)$. Then $\| e^{h} \| = e^{\rho(h)}$.

COROLLARY . Every Vidav algebra is *-isomorphic to a B*-algebra .

PROOF: The corollary will follow from Theorem 5 if we can show that every Vidav algebra is semi-simple.

Let A be a Vidav algebra with radical R . By Theorem 4, (2), $R^* = R$. Let $h \in R \cap H(A)$. By Theorem 6 , $\| e^{\alpha h} \| = 1$ for all $\alpha \in \mathbb{R}$, so that $ih \in H(A)$. Thus $h = 0$. The required result immediately follows .

Let A be a Vidav algebra . It is clear that the B*-norm $\|\cdot\|_0$ on A is given by $\|x\|_0^2 = \sup\{F(x^*x) : F \in D(1)\}$. Palmer has proved that in actual fact , if $\|\cdot\|$ is the given norm on A , $\|\cdot\| = \|\cdot\|_0$. This is an easy consequence of (0.1), Theorem 6 and the fact that, in a B*-algebra, the B*-norm is minimal . (See [2] .) This concludes the proof of the Vidav-Palmer theorem .

We conclude this section by examining a generalisation of the notion of a Vidav algebra in which the requirement that the algebra considered should contain an identity is dispensed with . This generalisation has essentially been studied independently by several authors . (See [6], §8 .)

DEFINITION 2 . A Banach algebra A is said to be a V-ALGEBRA if there exists a subset $H(A)$ of A with the following properties:

(a) $H(A) + iH(A) = A$.

(b) For each $a \in S(A)$ and each $h \in H(A)$,

$$\| a + i\alpha ha \| \leq 1 + o(\alpha) \qquad (\alpha \in \mathbb{R}, \alpha \to 0 .)$$

It is clear that every Vidav algebra is a V-algebra . It is easy to construct radical V-algebras ; in fact, if A is any Banach space and $H(A)$ is any subset of A such that $H(A) + iH(A) = A$, then A with the zero multiplication is a V-algebra . However, if such trivial cases are excluded , analogues of some of the results proved for Vidav algebras can be obtained .

It is easily verified that if A is a Vidav algebra , every element of $H(A)$ is a hermitian element of A . (If A is a V-algebra , $H(A)$ will be the given subset of A satisfying the conditions (a) and (b) of Definition 2 .)

THEOREM 7 . Let A be a V-algebra . Then :

(1) If R is the radical of A , $R = \{x \in A : xA = (0)\}$.

(2) If A is semi-simple, A is an A*-algebra with auxiliary norm $\|\cdot\|_1$, where $\|\cdot\|$ is the given norm on A ,

(3) A is a B*-algebra if and only if $v(\cdot)$ is equivalent to $\|\cdot\|$ on A .

PROOF: Let $a \to T_a$ be the left regular representation of A on A , $\hat{A} = \{T_a : a \in A\}$ and $H(\hat{A}) = \{T_a : a \in H(A)\}$. Then $H(\hat{A})$ consists of hermitian operators on A , and $\hat{A} = H(\hat{A}) + iH(\hat{A})$.

Let $\hat{A}^-$ be the completion of $\hat{A}$ in $B(A)$. By (0.1), Theorem 2 , $\hat{A}^- = H(\hat{A})^- + iH(\hat{A})^-$, where we are using an obvious notation. Let $\hat{B} = \hat{A}^{-} + \mathbb{C} I$, where I is the identity operator on A. Then, under the operator norm, $\hat{B}$ is a Vidav algebra, whose set of hermitian elements is $H(\hat{A})^{-} + \mathbb{R} I$. Application of the Vidav-Palmer theorem now easily shows that, if A is semi-simple, A is an A^*-algebra with auxiliary norm $\|\cdot\|_1$. This proves (2).

If A is not necessarily semi-simple, $\|\cdot\|_1$ is an algebra semi-norm on A, and $K = \{x \in A : xA = (0)\} = \{y \in A : \|y\|_1 = 0\}$. Since A/K is clearly an A^*-algebra, it is semi-simple. Thus K is the radical of A.

(3) is an obvious consequence of (0.1), Theorem 3.

NOTES. (1) (2) of Theorem 7 is contained in [6], Theorem 8.4.

(2) It would be interesting to know when an A^*-algebra with norm $\|\cdot\|$ and auxiliary norm $\|\cdot\|_1$ is a V-algebra.

(3) An example in [6], §8 shows that there are A^*-algebras of the type described in (2) which are not B^*-algebras.

(4.2) β^*-ALGEBRAS .

In (4.1) , we investigated Vidav's metrical characterisation of B^*-algebras with identity . Motivated by Vidav's results, we might enquire into other metrical characterisation of B^*-algebras. (Some such metrical characterisations are given in [6], § 7 .)

In this section, we examine those algebras which satisfy a certain metrical condition, the latter condition being automatically satisfied by all B^*-algebras ; we call any algebra satisfying the above condition a β^*-algebra . It is shown that every β^*-algebra with identity element is a B^*-algebra . Further, every commutative β^*-algebra is a B^*-algebra, though we have been unable to prove such a result for a general β^*-algebra .

We now define the notion of a β^*-algebra .

DEFINITION . Let A be a Banach $(\times, \circ, \#)$-algebra* with anti-linear involution $x \to x^*$. Then A is a β^*-ALGEBRA if

$$\| x \times x^* + x \circ x^* \| = 2 \|x\|^2 \quad \text{for all } x \in A .$$

We now note certain elementary properties of a β^*-algebra A .

(1) If "$\times$" = "$\circ$" , A is a B^*-algebra .

(2) If $x \in A$, $\|x\| = \|x^*\|$ and $\|x \times x^*\| = \|x\|^2 = \|x \circ x^*\|$.

(3) If A has an identity element 1 , $\|1\| = 1$.

Throughout this section, A is a β^*-algebra . Using the notations of (3.1) , $\rho_\times(x) = |Sp_\times(x)|$ and $\rho_\circ(x) = |Sp_\circ(x)|$ for each $x \in A$. For $x \in A$ and $n \in N$, we use the notation $x^n_\times$ (resp. $x^n_\circ$) to denote the $n^{th.}$ power of x for the multiplica-

* See p. 48 . For convenience , we denote the involution of A by $x \to x^*$ instead of by $x \to x^\#$.

tion "$\times$" (resp. "$\circ$") . $H(A) = \{x \in A ; x^* = x\}$.

LEMMA 1 . Let $h \in H(A)$. Then $\rho_\times(h) = \|h\| = \rho_\circ(h)$.

PROOF: Let $h \in H(A)$. We define by recursion the following sequence $\{a_n\}$ in A , constructed as follows :

$$a_1 = h \times h + h \circ h \text{ and for each } n \in N ,$$

$$a_{n+1} = a_n \times a_n^* + a_n \circ a_n^* .$$

For $n \in N$, let $N_n = 2^{(2^n - 1)}$. We note that $N_{n+1} = 2(N_n)^2$ for each $n \in N$. $\{a_n\}$ has the following properties .

(1) $\|a_n\| = N_n \|h\|^{2^n}$ for each $n \in N$. This result is clearly true when $n = 1$. If $k \in N$ and $\|a_k\| = N_k \|h\|^{2^k}$, then $\|a_{k+1}\| = 2\|a_k\|^2 = 2(N_k)^2 \|h\|^{2^{k+1}} = N_{k+1} \|h\|^{2^{k+1}}$.

The result now follows by induction .

(2) For each $n \in N$, let $b_n = a_n - h_\times^{2^n}$ and $c_n = a_n - h_\circ^{2^n}$. Then $\|b_n\| \leq (N_n - 1)\|h\|^{2^n}$ and $\|c_n\| \leq (N_n - 1)\|h\|^{2^n}$ for each $n \in N$.

This result is clearly true for $n = 1$. Suppose the result is true for $n = k$. Then

$$\begin{aligned} \|b_{k+1}\| &= \|a_{k+1} - h_\times^{2^{(k+1)}}\| \\ &= \|(h_\times^{2^k} + b_k) \times (h_\times^{2^k} + c_k^*) + a_k \circ a_k^* - h_\times^{2^{(k+1)}}\| \\ &\leq \|b_k\| . \|h\|^{2^k} + \|c_k\| \|h\|^{2^k} + \|b_k\| \|c_k\| + (N_k)^2 \|h\|^{2^{(k+1)}} \\ &\leq \|h\|^{2^{(k+1)}} ((N_k - 1)^2 + 2(N_k - 1) + (N_k)^2) \\ &= (N_{k+1} - 1) \|h\|^{2^{(k+1)}} . \end{aligned}$$

The corresponding result for $\|c_{k+1}\|$ may similarly be verified . The required result now follows by induction .

Now, let $n \in N$. Then

$$N_n \|h\|^{2^n} = \|a_n\| \leq \|a_n - h_x^{2^n}\| + \|h_x^{2^n}\|$$
$$\leq (N_n - 1)\|h\|^{2^n} + \|h_x^{2^n}\| .$$

Hence $\|h\|^{2^n} \leq \|h_x^{2^n}\|$, so that $\|h_x^{2^n}\| = \|h\|^{2^n}$.

It clearly follows that $\rho_x(h) = \|h\|$. A similar arguement shows that $\rho_o(h) = \|h\|$.

LEMMA 2. Suppose that A has an identity element 1. Let $h \in H(A)$ and suppose that $f \in D(1)$ such that $|f(h)| = \|h\|$. Then $f(h) = \pm\|h\|$.

PROOF: (We note that since $\|h\| \leq \rho(h) \leq v(h) \leq \|h\|$, there always exists $f \in D(1)$ such that $|f(h)| = \|h\|$.)

We may obviously suppose that $h \neq 0$. Let $\theta = \arg f(h)$. Then we have

$$f(1 + e^{-i\theta}h) = 1 + \|h\| = \|1 + e^{-i\theta}h\| . \text{ Hence}$$

$$2(1 + \|h\|)^2 = 2\|1 + e^{-i\theta}h\|^2$$
$$= \|(1 + e^{-i\theta}h) \times (1^* + e^{i\theta}h) + (1 + e^{-i\theta}h) \circ (1^* + e^{i\theta}h)\|$$
$$= \|1^* + e^{i\theta}h + e^{-i\theta}h \times 1^* + h \times h + 1 + e^{-i\theta}h + e^{i\theta}1 \circ h + h \circ h\|$$
$$\leq 2 + 2\|h\| + 2\|h\|^2 + |e^{i\theta} + e^{-i\theta}| . \|h\| .$$

Since $h \neq 0$, we have $|e^{i\theta} + e^{-i\theta}| = 2$, and hence $e^{i\theta} = \pm 1$. Thus $f(h) = \pm\|h\|$.

THEOREM 1. Let A be a β^*-algebra with identity 1. Then A is a B^*-algebra.

PROOF: We show firstly that $1 = 1^*$. Let $k = i(1^* - 1)$. Then $k \in H(A)$. Hence, by Lemma 2, there exists $f \in D(1)$ such that $f(k) = \pm\|k\|$, i.e. $if(1^*) - i = \pm\|k\|$. Hence

$f(1^*) = 1 \pm i\|k\|$. Since $\|1^*\| = 1$, $|f(1^*)| \leq 1$; hence $k = 0$. Thus $1^* = 1$. Finally, let $h \in H(A)$ and $\alpha \in \mathbb{R}$.

Then
$$\|1 + i\alpha h\|^2 = \|(1 + i\alpha h) \times (1 - i\alpha h)\|$$
$$= \|1 + \alpha^2 h \times h\|$$
$$\leq 1 + o(\alpha) \quad \text{as } \alpha \to 0 \ .$$

Hence A is a Vidav algebra and, using Lemma 1 and (4.1), Theorem 5, we see that A is a B^*-algebra with norm $\|\cdot\|$.

We now show that every commutative β^*-algebra is a B^*-algebra . In the following three lemmas, A will be assumed to be a commutative β^*-algebra . Φ_x will denote the carrier space of A for the multiplication "$\times$" *.

LEMMA 3 . For each $x \in A$, there exists $\phi \in \Phi_x$ such that $\widehat{x \times x^*}(\phi) = \widehat{x \circ x^*}(\phi) = \|x\|^2 e^{i\theta}$, where $\theta \in \mathbb{R}$.

PROOF: Let $x \in A$. By Lemma 1 , we have $\rho_x(x \times x^* + \widehat{x \circ x^*}) = 2\|x\|^2$. Hence there exists $\phi \in \Phi_x$ such that $|\widehat{x \times x^*}(\phi) + \widehat{x \circ x^*}(\phi)| = 2\|x\|^2$. Hence $\arg[\widehat{x \times x^*}(\phi)] = \arg[\widehat{x \circ x^*}(\phi)]$, and $|\widehat{x \times x^*}(\phi)| = |\widehat{x \circ x^*}(\phi)| = \|x\|^2$. The required result now follows .

LEMMA 4 . Let $h \in S(A) \cap H(A)$ and $f \in D(h)$. Then for each $h_1 \in H(A)$, $f(h_1) \in \mathbb{R}$.

PROOF: Let $h_1 \in H(A)$. Then for $\alpha \in \mathbb{R}$,
$$\|h + i\alpha h_1\|^2 = \|(h + i\alpha h_1) \times (h - i\alpha h_1)\| = \|h_x^2 + \alpha^2 h_{x1}^2\| .$$
Hence $\|h + i\alpha h_1\| \leq 1 + o(\alpha) \qquad (\alpha \in \mathbb{R}, \alpha \to 0)$,
so that by a familiar arguement , if $f \in D(h)$, $\mathrm{Re} f(ih_1) = 0$, so that $f(h_1) \in \mathbb{R}$.

* In the sequel , $x \to \hat{x}$ will denote the Gelfand representation of A for the multiplication "$\times$" .

LEMMA 5 . Let $h \in S(A) \cap H(A)$. Then there exists $\phi \in \Phi_x$ such that $|\phi(h)| = \|h\|$, and, for each $h_1 \in H(A)$, $\phi(h_1) \in \mathbb{R}$.

PROOF: By Lemma 3 there exists $\phi \in \Phi_x$ such that

$$\phi(h \times h) = \phi(h \circ h) = \|h\|^2 e^{i\theta} \quad \text{where } \theta \in \mathbb{R}.$$

Let $\gamma = \arg \phi(h)$. Then $e^{-i\gamma}\phi \in D(h)$ so that by Lemma 4 , $e^{-i\gamma}\phi(h \times h + h \circ h) \in \mathbb{R}$. Since $\phi(h \times h) = \phi(h \circ h)$, $e^{-i\gamma}\phi(h \times h) \in \mathbb{R}$. But $e^{-2i\gamma}\phi(h \times h) = (e^{-i\gamma}\phi(h))^2 \in \mathbb{R}$. Hence $e^{i\gamma} \in \mathbb{R}$ and the result follows from Lemma 4 .

THEOREM 2 . A commutative β^*-algebra A is a B*-algebra .

PROOF: Using Lemma 5 , we may select for each $h \in H(A)$ an element $\phi_h \in \Phi_x$ such that $|\phi_h(h)| = \|h\|$ and $\phi_h(h_1) \in \mathbb{R}$ for every $h_1 \in H(A)$. Let $C = \{\phi_h : h \in H(A)\}$. We note that if $k \in H(A)$ and $\phi(k) = 0$ for all $\phi \in C$, then $k = 0$.

Now let $h \in H(A)$ and suppose that $h \times h = h_1 + ih_2$ where $h_1, h_2 \in H(A)$. For $\phi \in C$, $\phi(h)$, $\phi(h_1)$ and $\phi(h_2)$ are real, and $\phi(h \times h) = (\phi(h))^2$. Thus $\phi(h_2) = 0$ for all $\phi \in C$. Hence $h \times h \in H(A)$ for all $h \in H(A)$. By (3.1), Theorem 1, (2) , it follows that "$\times$" = "$\circ$" , so that A is a B*-algebra .

The author has been unable to show that a general β^*-algebra is a B*-algebra . The following theorem, which is proved by using Theorem 2, (3.1), Theorem 3, Corollary and (3.1), Theorem 1, (2) was the best result which was obtained .

THEOREM 3 . Let A be a β^*-algebra satisfying the condition (δ) of (3.1) . Then A is a B*-algebra .

REFERENCES

1. H.F.Bohnenblust and S. Karlin, 'Geometrical properties of the unit sphere of Banach algebras', Ann. of Math. 62(1955), 217 - 219.

2. F.F. Bonsall, 'A minimal property of the norm in some Banach algebras', J. London Math. Soc. 29(1954), 156 - 164.

3. F.F. Bonsall, 'The numerical range of an element of a normed algebra', Glasgow Math. J. 10(1969), 68-72.

4. F.F. Bonsall, B.E. Cain and H. Schneider, 'The numerical range of a continuous mapping of a normed space', Aequationes Math. 2(1968), 86 - 93.

5. F.F. Bonsall and J. Duncan, 'Dual representations of Banach Algebras', Acta Math. 117 (1967), 79 - 102.

6. F.F. Bonsall and J. Duncan, 'Numerical Range', (to appear in "London Math. Soc. Lecture Note Series").

7. M. Finkelstein and A. Lebow, 'A note on "Nth roots of operators"', Proc. Amer. Math. Soc. 21 (1969), 250.

8. J.W.M. Ford, 'A square root lemma for Banach *-algebras', J. London Math. Soc. 42(1967), 521-522.

9. L.T. Gardner, 'Square roots in Banach algebras', Proc. Amer. Math. Soc 17(1966), 132-134.

10. A.M. Gleason, 'A characterisation of maximal ideals', J. Analyse Math. 19(1967), 171-172.

11. E. Hille, 'Functional analysis and semi-groups', Amer. Math. Soc. Coll. Publ. 31, New York (1948) .

12. E. Hille, 'On roots and logarithms of elements of a complex Banach algebra', Math. Ann. 136(1958), 46-57 .

13. B. E. Johnson, 'The uniqueness of the (complete) norm topology', Bull. Amer. Math. Soc. 73(1967), 537-539 .

14. R. V. Kadison, 'Isometries of operator algebras', Ann. of Math. 54(1951), 325-338 .

15. G. Lumer, 'Semi-inner-product-spaces', Trans. Amer. Math. Soc. 100(1961), 29-43 .

16. P. Miles, 'B*-algebra unit ball extremal points', Pac. J. Math. 14(1)(1964), 627-637 .

17. M. A. Naimark, 'Normed Rings', P. Noordhoff N.V., Groningen, 1959 .

18. T. W. Palmer, 'Characterizations of C*-algebras', Bull. Amer. Math. Soc. 74(1968), 538-540 .

19. A. L. T. Paterson, 'Isometries between B*-algebras', (to appear in Proc. Amer. Math. Soc.) .

20. C. E. Rickart, 'General theory of Banach algebras', Van Nostrand, Princeton, N.J., 1960 .

21. B. Russo and H. A. Dye, 'A note on unitary operations in C*-algebras', Duke Math. J., 33(1966), 413-416 .

22. S. Shirali and J. W. M. Ford, 'Symmetry in complex involutory Banach algebras II', (to appear in Duke Math. J.).

23. Minoru Tomita, 'The second dual of a C*-algebra', Mem. Fac. of Sci., Kyushu Univ., Ser. A 21(1967), 185-193 .

24. I. Vidav, 'Eine metrische Kennzeichnung der selbstadjungierten Operatoren', Math. Zeit. 66(1956), 121-128 .